SM315 Lecture Notes
Separation of Variables
Homework: (55) 1a, 2a, 3a

1. Separable Function – Examples

a. ***<u>Separable Function</u>*: When a function** $f(x,y)$ ***can be expressed as*** $g(x)h(y)$

b. $xy + x = x(y+1) \rightarrow separable$

c. $xy + x + y + 1 = (x+1)(y+1) \rightarrow separable$

d. $\sqrt{xy} = \sqrt{x}\sqrt{y} \rightarrow separable$

e. $\sqrt{x+y} \rightarrow non-separable$

2. Fourier and the Heat Equation

a. ***PDE:*** $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. ***BC:*** $u(0,t) = u(L,t) = 0$

c. ***IC:*** $u(x,0) = f(x)$

d. ***Classify:*** $u_t - ku_{xx} = 0 \rightarrow$***, 2nd order,*** $u(x,t)$***, linear, homogeneous, constant coefficients, parabolic (i.e.*** $B^2 - 4AC = 0$***)***

3. **Assume $u(x,t)$ is separable, i.e. $u(x,t)=\phi(t)G(x)\rightarrow u=\phi G$**

   a. ***Therefore*** $u_t=\frac{d\phi}{dt}G=\phi' G$ ***and*** $u_{xx}=\phi\frac{d^2G}{dx^2}=\phi G''$

   b. ***Rewrite PDE as*** $u_t=ku_{xx}\rightarrow\phi' G=k\phi G''$

   c. ***Divide both sides by*** $\phi G\rightarrow\frac{\phi' G}{k\phi G}=\frac{\phi G''}{\phi G}\rightarrow\frac{1}{k}\frac{\phi'}{\phi}=\frac{G''}{G}=-\lambda$

      i. Note: Carry k with ϕ to follow Fourier's solution
      ii. Note: λ must be a constant and is called an eigenvalue
      iii. Note: Used $-\lambda$, again to follow Fourier's solution

4. **Now we can generate two differential equations that are solvable.**

   a. ***DE in time:*** $\frac{1}{k}\frac{\phi'}{\phi}=-\lambda\rightarrow\phi'=-k\lambda\phi\rightarrow\phi'+k\lambda\phi=0$

   b. ***DE in space:*** $\frac{G''}{G}=-\lambda\rightarrow G''=-\lambda G\rightarrow G''+\lambda G=0$

5. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ ***thus:*** $G'' - \lambda G = 0 \rightarrow G = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

i. BC: $G(0) = c_1 + c_2 = 0$ ***and*** $G(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}$

ii. This implies $c_1 = c_2 = 0 \rightarrow G(x) = 0 \rightarrow$ ***trivial***

b. Assume $\lambda = 0$ ***thus:*** $G'' = 0 \rightarrow G = c_1 x + c_2$

i. BC: $G(0) = c_2 = 0$ ***and*** $G(L) = c_1 L = 0 \rightarrow c_1 = 0$

ii. This implies $G(x) = 0 \rightarrow$ ***trivial***

c. Assume $\lambda < 0$ ***thus:*** $G'' + \lambda G = 0 \rightarrow G = c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right)$

i. BC: $G(0) = c_2 = 0$ ***and*** $G(L) = c_1 \sin\left(\sqrt{\lambda}L\right) = 0$

ii. This implies $\sin\left(\sqrt{\lambda}L\right) = 0 \rightarrow \sqrt{\lambda}L = n\pi$ ***where*** $n = 1,2,3,\ldots$

iii. This implies $\lambda = \left(\frac{n\pi}{L}\right)^2$ ***where*** $n = 1,2,3,\ldots$

iv. By Superposition: $G(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

6. Now solve for $\phi(t)$.

a. Recall $\phi' + k\lambda\phi = 0 \rightarrow \phi(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

7. Now put it all together to get $u(x,t)$

a. $u(x,t) = \phi(t)G(x) \rightarrow b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \rightarrow \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

8. Finally Apply Initial Condition

a. $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

b. Fourier's Dilemma …

c. Example: Let $f(x) = 2\sin\left(\frac{3\pi x}{L}\right)$

i. Thus $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = 2\sin\left(\frac{3\pi x}{L}\right)$

ii. This implies $c_n = \begin{cases} 2 & for\ n = 3 \\ 0 & for\ n \neq 3 \end{cases}$

iii. Therefore $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) = 2e^{\frac{9k\pi^2 x}{L^2}} \sin\left(\frac{3\pi x}{L}\right)$