SM315 Lecture Notes
Orthogonal Functions/Fourier Series
Homework: (Handout)

Top Board

- ***Vectors:*** $\vec{a} \bullet \vec{b} = 0$
- ***We rewrite this in integral form as*** $\int_0^L f(x)g(x)dx = 0$
- ***Orthogonality of sin function for*** $0 < x < \pi$
  - $\int_0^\pi \sin(nx)\sin(mx)dx = \begin{cases} \pi/2 & for \quad n = m \\ 0 & for \quad n \neq m \end{cases}$
- ***Or more generally:***
  - $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2 & for \quad n = m \\ 0 & for \quad n \neq m \end{cases}$
- ***Can also show that:***
  - $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L & for \quad n = m \\ 0 & for \quad n \neq m \end{cases}$

1. **Recall result from last lecture when we applied initial condition:**
   - ***a.*** $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$
   - ***b. For Simplicity, Let*** $L = \pi \rightarrow \sum_{n=1}^{\infty} c_n \sin(nx) = f(x)$
   - ***c. To determine coefficients we take advantages of orthogonality properties of sines and cosines***

2. Definition of Orthagonality:

a. ***Vectors:*** $\vec{a} \bullet \vec{b} = 0$***, i.e.***

i. **2- D:** $\langle 1,1 \rangle \bullet \langle 1,-1 \rangle = 0$

ii. **3- D:** $\langle 1,1,1 \rangle \bullet \langle 1,1,-2 \rangle = 0$

iii. **4- D:** $\langle 1,1,1,1 \rangle \bullet \langle 1,1,1,-3 \rangle = 0$

b. ***Two Functions over a given length have an infinite number of points and form an infinite number of points.***

i. **Orthoganality of 2 Functions would be** $\sum_{i=1}^{\infty} f(x_i)g(x_i) = 0$ **for** $0 < x < L$

ii. **We rewrite this in integral form as** $\boxed{\int_0^L f(x)g(x)dx = 0}$

3. Orthogonalitiy of sines and cosines

a. ***Calc Demo: Pick 2 different positive integer n, m.***

i. **Integrate Calc:** $\int_0^{\pi} \sin(nx)\sin(mx)dx$ **- Should =0**

b. ***Now pick any positive integer n***

i. **Integrate Calc:** $\int_0^{\pi} \sin(nx)\sin(nx)dx$ **- Should =** $\pi/2$

c. ***It appears that*** $\sin(mx)$ ***is orthogonal to*** $\sin(nx)$ ***on*** $0 < x < \pi$ ***for*** $n \neq m$

4. Prove: $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$ for $n \neq m$

a. ***Note that this was the original form before letting*** $L = \pi$

b. ***Recall:***

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

c. Therefore:
$$\cos(a-b)-\cos(a+b)=2\sin(a)\sin(b)\rightarrow$$
$$\sin(a)\sin(b)=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$$

d. Hence: $\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)=\frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right)-\cos\left(\frac{(m+n)\pi x}{L}\right)\right]$

e. And: $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx=\int_o^L \frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right)-\cos\left(\frac{(m+n)\pi x}{L}\right)\right]dx$

f. 1st Integral:
$$\frac{1}{2}\int_o^L \cos\left(\frac{(m-n)\pi x}{L}\right)dx=\frac{L}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right)\Bigg|_0^L=$$
$$K(\sin(m-n)\pi-\sin(0))=0$$

g. 2nd Integral:
$$\frac{1}{2}\int_o^L \cos\left(\frac{(m+n)\pi x}{L}\right)dx=\frac{L}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\Bigg|_0^L=$$
$$K(\sin(m+n)\pi-\sin(0))=0$$

h. Therefore: $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$ **=0 for** $n\neq m$

5. Homework Examples