SM315 Lecture Notes
Convergence of Fourier Series
Homework: (95)1abc, 2ab

Top Board

- ***We have shown:***
  - $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L & for \quad n = m \\ 0 & for \quad n \neq m \end{cases}$
  - $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L & for \quad n = m \\ 0 & for \quad n \neq m \end{cases}$
  - $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$

1. Recall Fourier's Dilemma when he applied initial conditions to the heat eqaution:

a. $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$ →How does one determine the coefficients c_n

b. Multiply both sides by $\sin\left(\frac{m\pi x}{L}\right) \rightarrow \sum_{n=1}^{\infty} c_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = f(x)\sin\left(\frac{m\pi x}{L}\right)$

c. Integrate: $\int_{-L}^{L}\sum_{n=1}^{\infty} c_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$

d. Left hand side rewritten: $\sum_{n=1}^{\infty} c_n \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

e. All terms where $n \neq m$ are equal to zero.

f. All terms where $n = m = c_m L \rightarrow c_m = \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$

2. Example: find a sine series for $f(x) = x$ and $L = 1$ (MATLAB Demo – *go1*):

a. $c_n = \int_{-1}^{1} x\sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2} sin(n\pi) = -\frac{2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi} = \frac{2}{\pi}\frac{(-1)^{n+1}}{n}$

b. i.e. $x \approx \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) = \frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x)...\right)$

c. Demonstration – note the periodic extension and the behavior at jumps

3. General Fourier Series over Interval $-L$ to L (p. 91)

a.
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

4. Convergence Theorem for Fourier Series (p. 92)

If $f(x)$ is *piecewise smooth* on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges

3. to the *periodic extension* of $f(x)$, were the periodic extension is continuous;
4. to the average of the two limits, usually $\frac{1}{2}\left[f(x+) + f(x-)\right]$ where the periodic extension has a *jump discontinuity*.

5. Example: find a Fourier Series for $f(x) = \begin{cases} 1 & -2 \le x \le 0 \\ x & 0 < x \le 2 \end{cases}$ (MATLAB Demo – *go2*):

a.
$$a_0 = \frac{1}{4}\left[\int_{-2}^{0} dx + \int_{0}^{1} x\,dx\right]$$
$$a_n = \frac{1}{2}\left[\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{1} x\cos\left(\frac{n\pi x}{2}\right)dx\right]$$
$$a_n = \frac{1}{2}\left[\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{1} x\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

b. Do integrals on calculator or Maple

c. Matlab Demo go2

d. Why did I need sines and cosines to do this expansion when before I only need sines?????

6. Sketching a Fourier Series

a. Sketch $f(x)$ on interval $-L < x < L$
b. Sketch the periodic extension
c. Mark an ‘x’ at the average for any jump discontinuities.