SM315 Lecture Notes
Fourier Sine/Cosine Series
Homework: (114)1abc, 2a,6a

<u>General Fourier Series</u>

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Fourier Sine Series (odd functions)</u>

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Fourier Cosine Series (even functions)</u>

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

1. Recall that function $f(x) = x$ required only sines to create a Fourier Series.

a. For odd functions $f(x) = -f(x)$ Fourier Series Simplifies

i. $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L}\{odd\ \ function\}dx = 0$

ii. $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}\{odd\ \ function\}dx = 0$

1. Note (odd function)*(even function)=odd function
2. Note: cos expressions eliminated for odd functions because cos is an even function

iii. $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}\{even\ \ function\}dx$

$$= \frac{2}{L}\int_{0}^{L}\{even\ \ function\}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

b. For even functions $f(x) = f(-x)$ Fourier Series Simplifies

i. $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{L}\{even\ \ function\}dx = \frac{1}{L}\int_{0}^{L} f(x)dx$

ii. $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}\{even\ function\}dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

1. Note (even function)*(even function)=even function

iii. $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}\{odd\ \ function\}dx = 0$

1. Note: sin terms eliminated for even functions

2. Calc Example: Find a cosine series for $f(x) = x^2$ and $0 \le x \le \pi$

a. $a_0 = \frac{1}{\pi}\int_0^\pi x^2 dx = \frac{1}{\pi}\frac{1}{3}\pi^3 = \frac{\pi^2}{3}$

b. $a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos(nx) dx = \frac{4}{n^2}\cos(n\pi) + \frac{2\pi}{n}\sin(n\pi) - \frac{4}{n^3\pi}\sin(n\pi)$

   i. Note: $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$

c. Therefore: $a_n = \frac{4}{n^2}(-1)^n$

d. $x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$

3. Calc Example: Find a sine series for $f(x) = x^2$ and $0 \le x \le \pi$

a. $b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin(nx) dx = -\frac{2\pi}{n}\cos(n\pi) + \frac{4}{n^3\pi}\cos(n\pi) + \frac{4}{n^2}\sin(n\pi) - \frac{4}{n^3\pi}$

b. $b_n = -\frac{2\pi}{n}(-1)^n + \frac{4}{n^3\pi}(-1)^n - \frac{4}{n^3\pi} = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left((-1)^n - 1\right)$

c. Or: $b_n = \begin{cases} \frac{2\pi}{n} - \frac{8}{n^3\pi} & for\ n = 1,3,5,\ldots \\ -\frac{2\pi}{n} & for\ n = 1,3,5,\ldots \end{cases}$

d. $x^2 \sim \sum_{n=1}^{\infty} b_n \sin(nx)$

4. Demo: Plot Above cosine and sine series (MATLAB-*Demo*)

a. Note odd and even extensions.