SM315 Lecture Notes
Term-by-Term Differentiation of Fourier Series Homework: (125)4ab, 5

Theorem 1: Term-by-Term Differentiation of Fourier Series

If [1] the Fourier series of $f(x)$ on the interval $-L \le x \le L$ is continuous from $(-\infty, \infty)$, and [2] $f'(x)$ and $f''(x)$ are piecewise continuous, then the Fourier series of $f'(x)$ can be obtained by term-by-term differentiation of the Fourier series of $f(x)$.

- Note, this is only true if $f(-L) = f(L)$
- Note $f'(x)$ does not exist at jump discontinuites

Theorem 2: Term-by-Term Differentiation of Sine Series

If [1] the sine series of $f(x)$ on the interval $0 \le x \le L$ is continuous from $(-\infty, \infty)$, and [2] $f'(x)$ and $f''(x)$ are piecewise continuous, then the sine series of $f'(x)$ can be obtained by term-by-term differentiation of the sine series of $f(x)$.

- Note, this is only true if $f(0) = f(L)$ … otherwise $f(x)$ is not continuous

Theorem 3: Term-by-Term Differentiation of Cosine Series

If [1] the cosine series of $f(x)$ on the interval $0 \le x \le L$ is continuous from $(-\infty, \infty)$, and [2] $f'(x)$ and $f''(x)$ are piecewise continuous, then the cosine series of $f'(x)$ can be obtained by term-by-term differentiation of the cosine series of $f(x)$.

- Note, a stipulation for $f(0)$ & $f(L)$ is not needed (why?).

1. Examples Demonstrating (but not Proving) Theorem 1.

a. Consider $f(x) = x^3$ for $-L \le x \le L$. The Fourier series for $f'(x) = 3x^2$ **can not** be determined by term-by-term differentiation of the Fourier series for $f(x)$ … why?

b. Consider $f(x) = x$ for $-L \le x \le L$. The Fourier series for $f'(x) = 1$ **can not** be determined by term-by-term differentiation of the Fourier series for $f(x)$.

   i. Recall $x \sim \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)$

      1. use calc if necessary i.e. $\frac{1}{L} \int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx$

   ii. Take derivative of both sides: $1 \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{n\pi x}{L}\right)$

   iii. But the cosine expansion of 1 is simply $a_0 = 1$

c. However, Consider $f(x) = x^2$ for $-L \le x \le L$. The Fourier series for $f'(x) = 2x^2$ **can** be determined by term-by-term differentiation of the Fourier series for $f(x)$ … why?

2. Proof of Theorem 2.

a. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

b. Then $f'(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

c. Equate the expression in (b) to the term by term differentiation of (a):

   i. i.e. $f'(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

d. This implies that $a_n = \frac{n\pi}{L} b_n$ and that $a_0 = 0$

e. But $a_0 = \frac{1}{L}\int_0^L f'(x)dx = \frac{1}{L} f(x)\Big|_0^L = \frac{1}{L}\left[f(L) - f(0)\right] = 0$

f. This implies that $f(L) = f(0)$.

3. Discussion of Theorem 3.

a. For an even function $f(0) = f(L)$ is not a necessary stipulation to ensure that cosine series is continuous.

   i. Show with a graphical demonstration.