SM315 Lecture Notes
Term-by-Term Integration of Fourier Series
Homework: (131) 2ab

A Fourier series of a piecewise smooth function $f(x)$ can always be integrated term-by-term. The result is a convergent infinite series to the $\int f(x)dx$ for $-L \le x \le L$ (even if the Fourier series has jump discontinuities.

1. Example

a. Recall $x \sim \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$

b. This implies the $\int x dx = \frac{1}{2}x^2 \sim \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^{n+2}\cos\left(\frac{n\pi x}{L}\right) + C$

   i. $C = a_0 = \frac{1}{2}\int_{-1}^{1}\frac{1}{2}x^2 dx = \frac{1}{2}\int_{0}^{1}x^2 dx = \frac{1}{6}$

   ii. $\frac{1}{2}x^2 \sim \frac{1}{6} + \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^{n+2}\cos\left(\frac{n\pi x}{L}\right)$

   iii. Demo: go1→ let L=1

c. This implies the $\frac{1}{2}\int x^2 dx = \frac{1}{6}x^3 \sim \frac{1}{6}x + \frac{2L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(-1)^{n+1}\cos\left(\frac{n\pi x}{L}\right) + C$

   i. $C = a_0 = \frac{1}{2}\int_{-1}^{1}\frac{1}{6}x^3 dx = 0$

   ii. Continue Demo: go1→ let L=1

   iii. What happened to the periodicity property???