SM315 Lecture Notes
Derivation of the Heat Equation
Homework: (14) 1,2; (19) 1adfh,3

- **Heat Equation with a Source:**
  - $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{dx^2}$

1. Derivation Follows Below

2. Thermal Energy Density:

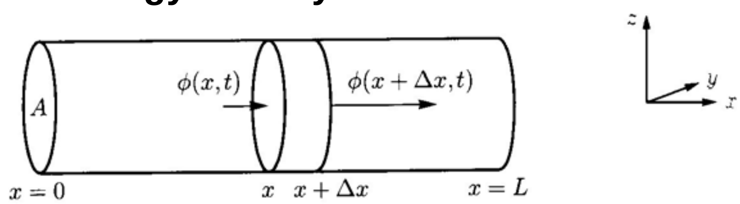

Figure 1.2.1 One-dimensional rod with heat energy flowing into and out of a thin slice.

a. Let $e(x,t)=$ thermal energy density
b. Assume thermal quantities are constant across a section
c. Assume the rod is one-dimensional
d. Assume rod is perfectly insulate on lateral surfaces
e. Assume thermal density varies from one cross section to next.

3. Total Heat Energy in a Cross Section:

a. Heat energy $= e(x,t)A\Delta x$

4. Conservation of Heat Energy:

Rate of change of heat energy in time = Heat energy flowing across the boundaries per unit time + Heat energy generated inside per unit time

a. Heat Flux: $\phi(x,t)$ amount of thermal energy *per unit time per unit area*
   i. $\phi(x,t) < 0$ → energy is flowing left
   ii. $\phi(x,t) > 0$ → energy is flowing right
b. Energy per unit time flowing across boundaries $= \phi(x,t)A - \phi(x+\Delta x,t)A$
c. Heat Source: $Q(x,t)$ = heat energy *per unit volume* generate *per unit time*
   i. Heat energy generated per unit time = $Q(x,t)A\Delta x$

d. Therefore, conservation of energy in a thin slice becomes:

$$\frac{\partial}{\partial t}[e(x,t)A\Delta x] \approx \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)A\Delta x \rightarrow$$

$$\frac{\partial e}{\partial t}A\Delta x \approx \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)A\Delta x \rightarrow$$

$$\frac{\partial e}{\partial t} \approx \frac{\phi(x,t)A - \phi(x+\Delta x,t)}{\Delta x} + Q(x,t)$$

e. Let $\Delta x \rightarrow 0 \rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t)$

5. Temperature and specific heat (links energy to temperature):

a. Let $\begin{cases} u(x,t) = temperature \\ c(x) = specific\ heat \\ \rho(x) = density\ \ (mass\ \ per\ unit\ volume) \end{cases}$

b. Recall specific heat is the energy that must be supplied to raise a unit mass of a substance one temperature unit.

c. Total mass of thin slice is $\rho(x)A\Delta x$

d. Total energy in thin slice = $e(x,t)A\Delta x = c(x)\rho(x)u(x,t)A\Delta x \rightarrow e(x,t) = c(x)\rho(x)$

e. Therefore $\begin{cases} \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t) \rightarrow \\ \frac{\partial}{\partial t}[c(x)\rho(x)u(x,t)] = -\frac{\partial \phi}{\partial x} + Q(x,t) \rightarrow \\ c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t) \end{cases}$

6. Fourier's Law of Heat Conduction (links flux to temperature):

a. $\phi = -K_0 \frac{\partial u}{\partial x} \rightarrow$ Where K_o is the thermal conductivity of the object.

b. Therefore: $c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[K_0 \frac{\partial u(x,t)}{\partial x}\right] + Q(x,t)$

7. Finally the Heat Equation:

a. Assume $c, \rho,$ and K_0 are constant

b. Let $Q = 0$ and $k = \frac{K_0}{c\rho}$

c. $c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \rightarrow$ **viola!**

8. Boundary Conditions:

a. Prescribed Temperature (Direchlet):

  i. i.e. $u(0,t) = u_0(t)$ and $u(L,t) = u_L(t)$

  ii. usually we let them be constant, i.e. $u(0,t) = T_0$ and $u(L,t) = T_L$

  iii. but so far we can only solve equations where: $u(0,t) = u(L,t) = 0$

b. Insulated Boundary (Neumann): $\frac{\partial u}{\partial x} = 0$

c. Newton's Law of Cooling (Mixed)

  i. $\frac{\partial u}{\partial x}(0,t) = K(u(0,t) - u_B(t)) \to \frac{\partial u}{\partial x} + Ku = -u_B(t)$

d. Note: Changing Boundary Conditions may effect eigenvalues

9. Equilibrium/Steady State Problem:

a. Implies $\frac{\partial u}{\partial t} = 0$, thus the heat equation becomes: $k\frac{d^2u}{dx^2} = 0 \to \frac{d^2u}{dx^2} = 0$

b. Solving would yield: $u(x) = c_1 x + c_2$

c. Example: Let $u(0,t) = 10$ and $u(L,t) = 100$

$$\to c_2 = 10 \to u(x) = c_1 x + 10 \to u(L) = c_1 L + 10 = 100 \to c_1 = \frac{90}{L}$$

$$\boxed{\to u(x) = \frac{90}{L}x + 10}$$