SM315 Lecture Notes
Heat Equation with $T=0$ Ends
Homework: (55) 1cf, 2bd

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t) = u(L,t) = 0$

c. IC: $u(x,0) = f(x)$

d. Classify: $u_t - ku_{xx} = 0 \rightarrow$, 2nd order, $u(x,t)$, linear, homogeneous, constant coefficients, parabolic (i.e. $B^2 - 4AC = 0$)

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(t)G(x) \rightarrow u = \phi G$

a. Therefore $u_t = \frac{d\phi}{dt}G = \phi' G$ and $u_{xx} = \phi\frac{d^2G}{dx^2} = \phi G''$

b. Rewrite PDE as $u_t = ku_{xx} \rightarrow \phi' G = k\phi G''$

c. Divide both sides by $\phi G \rightarrow \frac{\phi' G}{k\phi G} = \frac{\phi G''}{\phi G} \rightarrow \frac{1}{k}\frac{\phi'}{\phi} = \frac{G''}{G} = -\lambda$

- Note: Carry k with ϕ to follow Fourier's solution
- Note: λ must be a constant and is called an eigenvalue
- Note: Used $-\lambda$, again to follow Fourier's solution

3. Now we can generate two differential equations that are solvable.

a. DE in time: $\frac{1}{k}\frac{\phi'}{\phi} = -\lambda \rightarrow \phi' = -k\lambda\phi \rightarrow \phi' + k\lambda\phi = 0$

b. DE in space: $\frac{G''}{G} = -\lambda \rightarrow G'' = -\lambda G \rightarrow G'' + \lambda G = 0$

4. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ thus: $G'' - \lambda G = 0 \rightarrow G = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

- BC: $G(0) = c_1 + c_2 = 0$ and $G(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}$
- This implies $c_1 = c_2 = 0 \rightarrow G(x) = 0 \rightarrow$ trivial

b. Assume $\lambda = 0$ thus: $G'' = 0 \rightarrow G = c_1 x + c_2$

- BC: $G(0) = c_2 = 0$ and $G(L) = c_1 L = 0 \rightarrow c_1 = 0$
- This implies $G(x) = 0 \rightarrow$ trivial

c. Assume $\lambda < 0$ thus: $G'' + \lambda G = 0 \rightarrow G = c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right)$

- BC: $G(0) = c_2 = 0$ and $G(L) = c_1 \sin\left(\sqrt{\lambda}L\right) = 0$
- This implies $\sin\left(\sqrt{\lambda}L\right) = 0 \rightarrow \sqrt{\lambda}L = n\pi$ where $n = 1,2,3,\ldots$
- This implies $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n = 1,2,3,\ldots$
- By Superposition: $G(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

5. Now solve for $\phi(t)$.

a. Recall $\phi' + k\lambda\phi = 0 \rightarrow \phi(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

6. Now put it all together to get $u(x,t)$

a. $u(x,t) = \phi(t)G(x) \rightarrow b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \rightarrow \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

7. Finally Apply Initial Condition

a. $u(x,0)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

b. Fourier's Dilemma … no longer a dilemma because we now know:

- $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

8. Example

a. PDE: $\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t)=u(1,t)=0$

c. IC: $u(x,0)=(1-x)x^2$

d. General Solution applies with values $L=1$ and $k=5$ plugged in:

$$u(x,t)=\sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) = \boxed{\sum_{n=1}^{\infty} c_n e^{-5(n\pi)^2 t} \sin(n\pi x)}$$

- Note: eigen value analysis does not change because boundary conditions were the same.
- What parts of the solution decay the fastest?
- What is the equilibrium solution?

e. Apply initial condition: $u(x,0)=(1-x)x^2$

$$u(x,0)=\sum_{n=1}^{\infty} c_n e^{o} \sin(n\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

f. Now from Sine series we have:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x)dx = 2\int_0^1 x^2(1-x)\sin(n\pi x)dx$$

g. Plug this into TI:

$$c_n = -8\frac{\cos(n\pi)}{(n\pi)^3} - 2\frac{\sin(n\pi)}{(n\pi)^2} + 12\frac{\sin(n\pi)}{(n\pi)^4} - \frac{4}{(n\pi)^3}$$

h. Recall: $\sin(n\pi)=0$ and $\cos(n\pi)=(-1)^n \rightarrow$

$$c_n = -8\frac{(-1)^n}{(n\pi)^3} - \frac{4}{(n\pi)^3} = 8\frac{(-1)^{n+1}}{(n\pi)^3} - \frac{4}{(n\pi)^3} = \frac{4}{(n\pi)^3}\left(2(-1)^{n+1} - 1\right) \rightarrow$$

$$\boxed{c_n = \begin{cases} \dfrac{4}{(n\pi)^3} & \text{for } n \text{ odd} \\ -\dfrac{12}{(n\pi)^3} & \text{for } n \text{ even} \end{cases}}$$

i. Final Solution when expanded:

$$\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}-1}{n^3}e^{-5(n\pi)^2 t}\sin(n\pi x) =$$

$$\frac{4}{\pi^3}e^{-5\pi^2 t}\sin(\pi x) - \frac{3}{2\pi^3}e^{-20\pi^2 t}\sin(2\pi x) + \frac{4}{27\pi^3}e^{-45\pi^2 t}\sin(3\pi x)\ldots$$

9. Demo – Matlab – go1 – Animates above example