SM315 Lecture Notes
Example Heat Equation with *T=0* Ends
Homework: (55) 3b, 7a*b*

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t) = u(L,t) = 0$

c. IC: $u(x,0) = \begin{cases} 0 & for \ \ 0 \le x < 1 \\ 1 & for \ \ 1 \le x \le 2 \end{cases}$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(t)G(x) \rightarrow u = \phi G$

a. Therefore $u_t = \frac{d\phi}{dt}G = \phi' G$ and $u_{xx} = \phi\frac{d^2G}{dx^2} = \phi G''$

b. Rewrite PDE as $u_t = ku_{xx} \rightarrow \phi' G = k\phi G''$

c. Divide both sides by $\phi G \rightarrow \frac{\phi' G}{k\phi G} = \frac{\phi G''}{\phi G} \rightarrow \frac{1}{k}\frac{\phi'}{\phi} = \frac{G''}{G} = -\lambda$

d. DE in time: $\frac{1}{k}\frac{\phi'}{\phi} = -\lambda \rightarrow \phi' = -k\lambda\phi \rightarrow \phi' + k\lambda\phi = 0$

e. DE in space: $\frac{G''}{G} = -\lambda \rightarrow G'' = -\lambda G \rightarrow G'' + \lambda G = 0$

3. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ thus: $G'' - \lambda G = 0 \rightarrow G = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$
- BC: $G(0) = c_1 + c_2 = 0$ and $G(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}$
- This implies $c_1 = c_2 = 0 \rightarrow G(x) = 0 \rightarrow$ trivial

b. Assume $\lambda = 0$ thus: $G'' = 0 \rightarrow G = c_1 x + c_2$
- BC: $G(0) = c_2 = 0$ and $G(L) = c_1 L = 0 \rightarrow c_1 = 0$
- This implies $G(x) = 0 \rightarrow$ trivial

c. Assume $\lambda < 0$ thus: $G'' + \lambda G = 0 \rightarrow G = c_1 \sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)$
- BC: $G(0) = c_2 = 0$ and $G(2) = c_1\sin(2\sqrt{\lambda}) = 0$
- This implies $\sin(2\sqrt{\lambda}) = 0 \rightarrow 2\sqrt{\lambda} = n\pi$ where $n = 1,2,3,...$
- This implies $\lambda = \left(\frac{n\pi}{2}\right)^2$ where $n = 1,2,3,...$
- By Superposition: $G(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right)$

4. Now solve for $\phi(t)$.

a. Recall $\phi' + k\lambda\phi = 0 \rightarrow \phi(t) = b_1 e^{-\lambda t} = b_1 e^{-\left(\frac{n\pi}{2}\right)^2 t}$

5. Now put it all together to get $u(x,t)$

a. $u(x,t) = \phi(t)G(x) \rightarrow b_1 e^{-k\left(\frac{n\pi}{2}\right)^2 t} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right) \rightarrow \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)$

6. Finally Apply Initial Condition

a. $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right) = f(x)$

b. $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^1 (0)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{2}{2}\int_1^2 (1)\sin\left(\frac{n\pi x}{2}\right)dx$

$\rightarrow c_n = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_1^2 = -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

c. Note: $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & for \;\; n = odd \\ -1 & for \;\; n = 2,6,10,,, \\ 1 & for \;\; n = 4,8,12,... \end{cases}$

d. Therefore simplify $c_n = \frac{2}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

e. Therefore: $\boxed{u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[(-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right]\frac{1}{n} e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)}$

7. Demo – Matlab – go1 – Animates above example