SM315 Lecture Notes
Heat Equation Insulated Ends
Homework: (69) 1b, 2*

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

c. IC: $u(x,0) = f(x)$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(t)G(x) \to u = \phi G$

a. Therefore $u_t = \frac{d\phi}{dt}G = \phi' G$ and $u_{xx} = \phi\frac{d^2 G}{dx^2} = \phi G''$

b. Rewrite PDE as $u_t = ku_{xx} \to \phi' G = k\phi G''$

c. Divide both sides by $\phi G \to \frac{\phi' G}{k\phi G} = \frac{\phi G''}{\phi G} \to \frac{1}{k}\frac{\phi'}{\phi} = \frac{G''}{G} = -\lambda$

d. DE in time: $\frac{1}{k}\frac{\phi'}{\phi} = -\lambda \to \phi' = -k\lambda\phi \to \phi' + k\lambda\phi = 0$

e. DE in space: $\frac{G''}{G} = -\lambda \to G'' = -\lambda G \to G'' + \lambda G = 0$

3. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ thus: $G'' - \lambda G = 0 \to G = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \to G' = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

- BC: $G'(0) = c_1\sqrt{\lambda} - c_2\sqrt{\lambda} = 0 \to c_1 - c_2 = 0$
- BC: $G'(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$
    - This implies $c_1 = c_2 = 0 \to G(x) = 0 \to$ trivial

b. Assume $\lambda = 0$ thus: $G'' = 0 \to G = c_1 x + c_2 \to G' = c_1 = a_0$
- BC: $G'(0) = G'(L) = c_1 = 0$
- This implies $G(x) = c_2 \to$ <u>not trivial</u>!!!

c. Assume $\lambda < 0$ thus: $G'' + \lambda G = 0 \rightarrow G = c_1 \sin\left(\sqrt{\lambda}\, x\right) + c_2 \cos\left(\sqrt{\lambda}\, x\right)$

- $G' = c_1 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right) - c_2 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right)$
- BC: $G'(0) = c_1 = 0 \rightarrow G = c_2 \cos\left(\sqrt{\lambda} x\right)$
- BC: $G'(L) = -c_2 \sqrt{\lambda} \sin\left(\sqrt{\lambda} L\right) = 0$
- This implies $\sin\left(\sqrt{\lambda} L\right) = 0 \rightarrow \sqrt{\lambda} L = n\pi$ where $n = 1,2,3,\ldots$
- This implies $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n = 1,2,3,\ldots$
- By Superposition: $G(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$
  1. Note: I let $c_n \rightarrow a_n$
  2. Insulated ends gives us a cosine series

4. Now solve for $\phi(t)$.

a. Recall $\phi' + k\lambda\phi = 0 \rightarrow \phi(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

5. Now put it all together to get $u(x,t)$

a. $u(x,t) = \phi(t) G(x) \rightarrow a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$

- <u>Note</u>: for first term $\lambda = 0 \rightarrow e^{-k\lambda t} = 1$

6. Finally Apply Initial Condition

a. $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$

7. Example

a. PDE: $\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}$

b. BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$

c. IC: $u(x,0) = (1-x)x^2$

d. General Solution applies with values $L = 1$ and $k = 5$ plugged in:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) = \boxed{a_0 + \sum_{n=1}^{\infty} a_n e^{-5(n\pi)^2 t} \cos(n\pi x)}$$

- What parts of the solution decay the fastest?
- What is the equilibrium solution?

e. Apply initial condition: $u(x,0) = (1-x)x^2$

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n e^o \cos(n\pi x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = (1-x)x^2$$

f. Now from Cosine series we have:

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = \int_0^1 x^2(1-x)dx = \frac{1}{12}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x)dx = 2\int_0^1 x^2(1-x)\cos(n\pi x)dx$$

g. Plug this into TI:

$$a_n = -2\frac{\cos(n\pi)}{(n\pi)^2} + 12\frac{\cos(n\pi)}{(n\pi)^4} + 8\frac{\sin(n\pi)}{(n\pi)^3} - \frac{12}{(n\pi)^4} \rightarrow$$

$$a_n = -\frac{2}{n^2\pi^2}(-1)^n + \frac{12}{n^4\pi^4}(-1)^n - \frac{12}{(n\pi)^4} \rightarrow$$

$$a_n = \frac{2}{n^2\pi^2}(-1)^{n+1} + \frac{12}{n^4\pi^4}\left((-1)^n - 1\right) \rightarrow$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi^2} & for\ n\ even \\ \frac{2}{n^2\pi^2} - \frac{24}{n^4\pi^4} & for\ n\ odd \end{cases}$$

h. Therefore:

$$u(x,t) = \frac{1}{12} + \sum_{n=1}^{\infty} a_n e^{-5(n\pi)^2 t}\cos(n\pi x)$$

8. Demo – Matlab – go1 – Animates above example