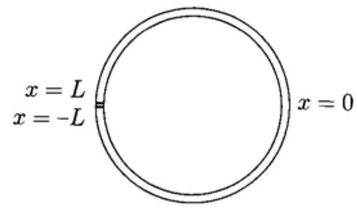
SM315 Lecture Notes
Heat Equation in a Circular Ring
Homework: (69) 3, 6

1. Heat Equation



a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. BC: $\begin{cases} u(-L,t) = u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) \end{cases}$

c. IC: $u(x,0) = f(x)$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(t)G(x) \rightarrow u = \phi G$

a. DE in time: $\frac{1}{k}\frac{\phi'}{\phi} = -\lambda \rightarrow \phi' = -k\lambda\phi \rightarrow \phi' + k\lambda\phi = 0$

b. DE in space: $\frac{G''}{G} = -\lambda \rightarrow G'' = -\lambda G \rightarrow G'' + \lambda G = 0$

3. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ thus:

$G'' - \lambda G = 0 \rightarrow G = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \rightarrow G' = c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}x}$

- BC: $\begin{cases} G(L) = G(-L) \rightarrow c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = c_1 e^{-\sqrt{\lambda}L} + c_2 e^{\sqrt{\lambda}L} \rightarrow \\ c_1\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) - c_2\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0 \rightarrow c_1 \sinh(L) - c_2 \sinh(L) = 0 \end{cases}$
- BC: $\begin{cases} G'(L) = G'(-L) \rightarrow c_1\sqrt{\lambda}e^{\sqrt{\lambda}L} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}L} = c_1\sqrt{\lambda}e^{-\sqrt{\lambda}L} - c_2\sqrt{\lambda}e^{\sqrt{\lambda}L} \rightarrow \\ c_1\sqrt{\lambda}\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) - c_2\sqrt{\lambda}\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0 \rightarrow c_1\sqrt{\lambda}\sinh(L) + c_2\sqrt{\lambda}\sinh(L) = 0 \end{cases}$
  - This implies $c_1 = c_2 = 0 \rightarrow G(x) = 0 \rightarrow$ trivial

b. Assume $\lambda = 0$ thus: $G'' = 0 \rightarrow G = c_1 x + c_2 \rightarrow G' = c_1$

- BC: $\begin{cases} G(L) = G(-L) = c_1 L + c_2 = -c_1 L + c_2 \rightarrow \\ c_1 = 0 \rightarrow G(x) = c_2 \rightarrow G'(x) = 0 \end{cases}$
  - BC: $G'(x) = 0$ satisfies the second conditions $G'(L) = G'(-L)$
  - This implies $G(x) = c_2 \rightarrow$ <u>not trivial</u>!!!
  - We will let $c_2 = a_0$

c. Assume $\lambda < 0$ thus: $G'' + \lambda G = 0 \to \begin{cases} G = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x) \\ G' = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} x) \end{cases}$

- BC: $\begin{cases} G(-L) = G(-L) = c_1 \sin(\sqrt{\lambda} L) + c_2 \cos(\sqrt{\lambda} L) = -c_1 \sin(\sqrt{\lambda} L) + c_2 \cos(\sqrt{\lambda} L) \to \\ c_1 \sin(\sqrt{\lambda} L) = 0 \to \lambda = \left(\frac{n\pi}{L}\right)^2 \end{cases}$
- BC: $\begin{cases} G'(-L) = G'(-L) \to \\ c_1 \sqrt{\lambda} \cos(\sqrt{\lambda} L) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda} L) + c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} L) \to \\ c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0 \to \lambda = \left(\frac{n\pi}{L}\right)^2 \end{cases}$
- Hence we can not eliminate either constant

d. Thus by superposition: $G = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L} x\right)$

4. Now solve for $\phi(t)$.

a. Recall $\phi' + k\lambda\phi = 0 \to \phi(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

5. Now put it all together to get $u(x,t)$

a. $u(x,t) = \phi(t) G(x) \to a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L} x\right)$

- Note: for first term $\lambda = 0 \to e^{-k\lambda t} = 1$

6. Finally Apply Initial Condition

a. $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L} x\right) = f(x)$

b. This is the standard Fourier series.

7. Example

a. PDE: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

b. BC: $\frac{\partial u}{\partial x}(-1,t) = \frac{\partial u}{\partial x}(1,t) = 0$

c. IC: $u(x,0) = 1 - x^2$

d. General Solution applies with values $L = 1$ and $k = 1$ plugged in:

- $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \cos(n\pi x) + \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$

- What parts of the solution decay the fastest?
- What is the equilibrium solution?
- Note: Since function is even, $\begin{cases} b_n = 0 \\ a_0 = \int_0^1 \left(1 - x^2\right) dx = \dfrac{2}{3} \\ a_n = 2\int_0^1 \left(1 - x^2\right)\cos(n\pi x)\, dx = -\dfrac{4(-1)^{n+1}}{n^2\pi^2} \end{cases}$
- MATLAB Demo “go1”

e. Apply initial condition: $u(x,0)=(1-x)x^2$

$$u(x,0)=a_0+\sum_{n=1}^{\infty}a_n e^o\cos(n\pi x)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\pi x)=(1-x)x^2$$

f. Now from Cosine series we have:

$$a_0=\frac{1}{L}\int_0^L f(x)dx=\int_0^1 x^2(1-x)dx=\frac{1}{12}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos(n\pi x)dx=2\int_0^1 x^2(1-x)\cos(n\pi x)dx$$

g. Plug this into TI:

$$a_n=-2\frac{\cos(n\pi)}{(n\pi)^2}+12\frac{\cos(n\pi)}{(n\pi)^4}+8\frac{\sin(n\pi)}{(n\pi)^3}-\frac{12}{(n\pi)^4}\rightarrow$$

$$a_n=-\frac{2}{n^2\pi^2}(-1)^n+\frac{12}{n^4\pi^4}(-1)^n-\frac{12}{(n\pi)^4}\rightarrow$$

$$a_n=\frac{2}{n^2\pi^2}(-1)^{n+1}+\frac{12}{n^4\pi^4}\left((-1)^n-1\right)\rightarrow$$

$$a_n=\begin{cases}-\dfrac{2}{n^2\pi^2} & \text{for } n \text{ even}\\[2ex] \dfrac{2}{n^2\pi^2}-\dfrac{24}{n^4\pi^4} & \text{for } n \text{ odd}\end{cases}$$

h. Therefore:

$$u(x,t)=\frac{1}{12}+\sum_{n=1}^{\infty}a_n e^{-5(n\pi)^2 t}\cos(n\pi x)$$

8. Demo – Matlab – go1 – Animates above example