SM315 Lecture Notes
Heat Equation with Non-Zero Temperature Ends
Homework: (352) 1abc

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

b. BC: $\begin{cases} u(0,t) = A \\ u(L,t) = B \end{cases}$

c. IC: $u(x,0) = f(x)$

2. Find Steady-State Solution

a. $\frac{\partial^2 u}{\partial x^2} = 0 \to u_E(x) = c_1 x + c_2$

b. Apply BC's
- $u(0) = c_2 = A \to u_E(x) = c_1 x + A$
- $u(L) = c_1 L + A = B \to c_1 = \frac{B-A}{L} \to u_E(x) = \left(\frac{B-A}{L}\right)x + A$

3. Set up New Function $v(x,t) = u(x,t) - u_E(x)$

a. $v(x,t) = u(x,t) - \left(\frac{B-A}{L}\right)x - A$
- Note: $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$
- Note: $\begin{cases} v(0,t) = u(0,t) - u_E(0) = A - A = 0 \\ v(L,t) = u(L,t) - u_E(L) = B - B = 0 \end{cases}$

b. Thus we can solve for this familiar PDE
- PDE: $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$
- BC: $\begin{cases} v(0,t) = 0 \\ v(L,t) = 0 \end{cases}$
- IC: $v(x,0) = f(x) - u_E(x) = f(x) - \left(\frac{B-A}{L}\right)x - A$

4. Let $v(x,t)=\phi(x)G(t)$ and Generate 2 Solvable ODEs.

a. DE in time: $\frac{1}{k}\frac{\phi'}{\phi}=-\lambda \rightarrow \phi'=-k\lambda\phi \rightarrow \phi'+k\lambda\phi=0$

b. DE in space: $\frac{G''}{G}=-\lambda \rightarrow G''=-\lambda G \rightarrow G''+\lambda G=0$

c. Eigenvalue analysis is the same for the $T=0$ BC's

- $\lambda \le 0$ implies the trivial solution
- $\lambda > 0$ thus implies $\lambda=\left(\frac{n\pi}{L}\right)^2$ and $G(x)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

d. $\phi'+k\lambda\phi=0 \rightarrow \phi(t)=b_1 e^{-\lambda t} = b_1 e^{-\left(\frac{n\pi}{L}\right)^2 t}$

e. $v(x,t)=\phi(t)G(x) \rightarrow \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

f. Apply Initial Condition: $$\left\langle \begin{array}{l} v(x,0)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)-\left(\frac{B-A}{L}\right)x - A \rightarrow \\ c_n = \frac{2}{L}\int_0^L \left(f(x)-\left(\frac{B-A}{L}\right)x-A\right)\left(\sin\left(\frac{n\pi x}{L}\right)\right)dx \end{array} \right.$$

g. Solve for $u(x,t)=v(x,t)+u_E(x)=\sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)+\left(\frac{B-A}{L}\right)x+A$

5. Example

a. PDE: $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

b. BC: $\begin{cases} u(0,t)=.5 \\ u(1,t)=1.5 \end{cases}$

c. IC: $u(x,0)=(1-x)x^2$

d. General Solution applies with values $L=1$, $k=1$, $A=.5$, and $B=1$ plugged in:

$$\left\langle \begin{array}{l} u(x,t)=\sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t} \sin(n\pi x)+x+.5 \rightarrow \\ c_n = 2\int_0^1 \left(-x^3+x^2-x-.5\right)\left(\sin(n\pi x)\right)dx \end{array} \right.$$

e. Plug this into TI:

$$\left\langle \begin{array}{l} c_n = 3\frac{\cos(n\pi)}{n\pi} - 8\frac{\cos(n\pi)}{(n\pi)^3} - 4\frac{\sin(n\pi)}{(n\pi)^2} + 12\frac{\sin(n\pi)}{n\pi} - \frac{1}{n\pi} - \frac{4}{(n\pi)^3} \rightarrow \\ c_n = 3\frac{(-1)^n}{n\pi} + 8\frac{(-1)^{n+1}}{(n\pi)^3} - \frac{1}{n\pi} - \frac{4}{(n\pi)^3} \end{array} \right.$$

$$\boxed{\rightarrow c_n = \begin{cases} -\frac{4}{n\pi} + \frac{4}{(n\pi)^3} & for\ n\ odd \\ \frac{2}{n\pi} - \frac{12}{(n\pi)^3} & for\ n\ even \end{cases}}$$

6. Demo – Matlab – go1 – Animates above example