SM315 Lecture Notes
Heat Equation with
Non-Zero Temperature Ends and Non-Zero Source
Homework: (352) 1df

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + D$ where D is a constant source

b. BC: $\begin{cases} u(0,t) = A \\ u(L,t) = B \end{cases}$

c. IC: $u(x,0) = f(x)$

2. Find Steady-State Solution

a. $\frac{\partial^2 u}{\partial x^2} = -\frac{D}{k} \to u_E(x) = -\frac{D}{2k}x^2 + c_1 x + c_2$

b. Apply BC's

- $u(0) = c_2 = A \to u_E(x) = -\frac{D}{2k}x^2 + c_1 x + A$
- $\left\langle \begin{array}{l} u(L) = -\frac{D}{2k}L^2 + c_1 L + A = B \to c_1 = \frac{B-A}{L} + \frac{DL}{2k} \to \\ u_E(x) = -\frac{D}{2k}x^2 + \left(\left(\frac{B-A}{L}\right) + \frac{DL}{2k}\right)x + A \end{array} \right.$

3. Set up New Function $v(x,t)=u(x,t)-u_E(x)$

a. $v(x,t)=u(x,t)+\frac{D}{2k}x^2-\left(\left(\frac{B-A}{L}\right)+\frac{DL}{2k}\right)x-A$

- Note: $\frac{\partial v}{\partial t}=\frac{\partial u}{\partial t}$ and $\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 u}{\partial x^2}+D$
- Note: $\begin{cases} v(0,t)=u(0,t)-u_E(0)=A-A=0 \\ v(L,t)=u(L,t)-u_E(L)=B-B=0 \end{cases}$

b. Thus we can solve for this familiar PDE

- PDE: $\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}$
- BC: $\begin{cases} v(0,t)=0 \\ v(L,t)=0 \end{cases}$
- IC: $v(x,0)=f(x)-u_E(x)=f(x)+\frac{D}{2k}x^2-\left(\left(\frac{B-A}{L}\right)+\frac{DL}{2k}\right)x-A$

4. Let $v(x,t)=\phi(x)G(t)$ and Generate 2 Solvable ODEs.

a. DE in time: $\frac{1}{k}\frac{\phi'}{\phi}=-\lambda\rightarrow\phi'=-k\lambda\phi\rightarrow\phi'+k\lambda\phi=0$

b. DE in space: $\frac{G''}{G}=-\lambda\rightarrow G''=-\lambda G\rightarrow G''+\lambda G=0$

c. Eigenvalue analysis is the same for the $T=0$ BC's

- $\lambda\leq 0$ implies the trivial solution
- $\lambda>0$ thus implies $\lambda=\left(\frac{n\pi}{L}\right)^2$ and $G(x)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{L}\right)$

d. $\phi'+k\lambda\phi=0\rightarrow\phi(t)=b_1e^{-\lambda t}=b_1e^{-\left(\frac{n\pi}{L}\right)^2 t}$

e. $v(x,t)=\phi(t)G(x)\rightarrow\sum_{n=1}^{\infty}c_ne^{-\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$

f. Apply Initial Condition:

$$\left\langle\begin{array}{l} v(x,0)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{L}\right)=f(x)+\frac{D}{2k}x^2-\left(\left(\frac{B-A}{L}\right)+\frac{DL}{2k}\right)x-A\rightarrow \\ c_n=\frac{2}{L}\int_0^L\left(f(x)+\frac{D}{2k}x^2-\left(\left(\frac{B-A}{L}\right)+\frac{DL}{2k}\right)x-A\right)\left(\sin\left(\frac{n\pi x}{L}\right)\right)dx \end{array}\right.$$

g. Solve for

$$u(x,t)=v(x,t)+u_E(x)=\sum_{n=1}^{\infty}c_ne^{-\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)-\frac{D}{2k}x^2+\left(\left(\frac{B-A}{L}\right)+\frac{DL}{2k}\right)x+A$$

5. Example

a. PDE: $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+1$

b. BC: $\begin{cases} u(0,t)=.5 \\ u(1,t)=1.5 \end{cases}$

c. IC: $u(x,0)=(1-x)x^2$

d. General Solution applies with values $L=1$, $k=1$, $A=.5$, $B=1.5$, and $D=1$ plugged in:

$$\left\langle \begin{array}{l} u_E(x)=-\frac{1}{2}x^2+\frac{3}{2}x+.5 \\ u(x,t)=v(x,t)+u_E(x)=\sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t}\sin(n\pi x)-\frac{1}{2}x^2+\frac{3}{2}x+.5 \rightarrow \\ c_n=2\int_0^1 (f(x)-u_E(x))\left(\sin\left(\frac{n\pi x}{L}\right)\right)dx=2\int_0^1(-x^3+1.5x^2-1.5x-.5)(\sin(n\pi x))dx \end{array}\right.$$

e. Plug equation for c_n this into TI (ignore $\sin(n\pi)$ terms):

$$\left\langle \begin{array}{l} c_n=3\frac{\cos(n\pi)}{n\pi}-6\frac{\cos(n\pi)}{(n\pi)^3}-\frac{1}{n\pi}-\frac{6}{(n\pi)^3} \rightarrow \\ c_n=3\frac{(-1)^n}{n\pi}+6\frac{(-1)^{n+1}}{(n\pi)^3}-\frac{1}{n\pi}-\frac{6}{(n\pi)^3} \end{array}\right.$$

$$\boxed{\rightarrow c_n=\begin{cases} -\frac{4}{n\pi} & \text{for } n \text{ odd} \\ \frac{2}{n\pi}-\frac{12}{(n\pi)^3} & \text{for } n \text{ even} \end{cases}}$$

6. Demo – Matlab – go1 – Animates above example

7. Demo – Matlab – go 2– Animates Example Below with Sink vice Source:

a. PDE: $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ -1

b. BC: $\begin{cases} u(0,t)=.5 \\ u(1,t)=1.5 \end{cases}$

c. IC: $u(x,0)=(1-x)x^2$