SM315 Lecture Notes
Day 22 – Heat Equation in Dimensions >1
Laplaces Equation on a Rectangle
MATLAB Project

1. Heat Equation in Multi-Dimensions

a. PDE: $\frac{\partial u}{\partial t} = k\nabla^2 u$ or in short form: $u_t = k\nabla^2 u$

b. $\nabla^2 u$ is called the "Laplacian of u"

1. 2D Rectangular (x, y): $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$
2. 3D Rectangular $(x, y.z)$: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u_{xx} + u_{yy} + u_{zz}$
3. 2D Polar (r, θ): $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$
4. 3D Cylindrical (r, θ, z): $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$
5. 3D Spherical (ρ, θ, ϕ): $\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta) + \frac{1}{\rho^2 \sin\theta}\frac{\partial^2 u}{\partial \phi^2}$

2. Steady State on a Rectangular Slab

a. $u_t = u_{xx} + u_{yy}$.

b. Recall: At steady state $u_t = 0 \rightarrow u_{xx} + u_{yy} = 0$ or simply $\nabla^2 u = 0$

c. $\nabla^2 u = 0$ is called LaPlace's Equation

d. Solve by Separation of Variables:
   - Let $u(x, y) = X(x)Y(y)$ or simply $u = XY$
   - Thus $u_{xx} + u_{yy} = 0$ can be rewritten as $X''Y + XY'' = 0$
   - This implies that $X''Y = -XY'' \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$
   - Thus we have two ordinary DE's: $\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$

3. Example Problem

1. $\left\langle \begin{array}{l} PDE: \ u_{xx} + u_{yy} = 0 \\ BC1: \ u(0,y) = g(y) \\ BC2: \ u(L,y) = 0 \\ BC2: \ u(x,0) = 0 \\ BC2: \ u(x,H) = 0 \end{array} \right.$ →

> **Recall 2nd Order Linear PDE w/2 Variables**
>
> $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$
>
> - **Parabolic**: $B^2 - 4AC = 0$
> - **Elliptic**: $B^2 - 4AC < 0$
> - **Hyperbolic**: $B^2 - 4AC > 0$

2. By Separation of variables we get $\begin{cases} X'' - \lambda X = 0 \ \ where \begin{cases} X(0) = g(x) \\ X(L) = 0 \end{cases} \\ Y'' + \lambda Y = 0 \ where \begin{cases} Y(0) = 0 \\ Y(H) = 0 \end{cases} \end{cases}$

3. Solve for Y first since it has the zero boundary conditions:
   - From previous analysis we know that: $\lambda > 0$ will produce the only non trivial solution with $\lambda = \left(\frac{n\pi}{H}\right)^2$ and $y = c_n \sin\left(\frac{n\pi}{H}\right)$

4. Now solve for $X \rightarrow X'' - \left(\frac{n\pi}{H}\right)^2 X = 0 \rightarrow X = b_1 \sinh\left(\frac{n\pi}{H}x\right) + b_2 \cosh\left(\frac{n\pi}{H}x\right)$

5. A note on sinh and cosh functions:
   - $\sinh(x) = \frac{e^x - e^{-x}}{2} \rightarrow \frac{d}{dx}(\sinh(x)) = \frac{e^x + e^{-x}}{2} = \cosh(x) \rightarrow \frac{d}{dx}(\cosh(x)) = \frac{e^x - e^{-x}}{2} = \sinh(x)$
   - Therefore the solution to say $x'' - 1 = 0$ can be expressed as $x = a_1 e^t + a_2 e^{-t}$ or $x = b_1 \sinh(t) + b_2 \cosh(t) = b_1\left(\frac{e^t - e^{-t}}{2}\right) + b_2\left(\frac{e^t + e^{-t}}{2}\right) = \left(\frac{b_1 + b_2}{2}\right)e^t + \left(\frac{b_2 - b_1}{2}\right)e^{-t}$
   - Experience of past mathematicians tells me to go with sinh/cosh

6. Experience also teaches us that the problem will be easier to solve if I shift the y-axis L units to the right. The problem is still the same as long as the shift is represented in the arguments: i.e. $X = b_n \sinh\left(\frac{n\pi}{H}(x - L)\right) + b_n \cosh\left(\frac{n\pi}{H}(x - L)\right)$

7. Now apply X boundary condition

8. $X(L) = b_1 \sinh(0) + b_2 \cosh(0) = b_2 = 0 \rightarrow X = b_n \sinh\left(\frac{n\pi}{H}(x-L)\right)$

9. We can now write $\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left(\frac{n\pi}{H}(x-L)\right)}$
   - Note: $A_n = b_n c_n$ and superposition principle was applied

10. Finally we apply non-zero boundary condition:
    - $u(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left(-\frac{L}{H} n\pi\right) = g(y) \rightarrow$
    - $\sum_{n=1}^{\infty} A_n \left[\int_0^H \left(\sin\left(\frac{m\pi}{H} y\right) \sin\left(\frac{n\pi}{H} y\right)\right) dy\right] \sinh\left(-\frac{L}{H} n\pi\right) = \int_0^H \left(g(y) \sin\left(\frac{m\pi}{H} y\right)\right) dy$
    - Thus: $A_m \left[\frac{H}{2}\right] \sinh\left(-\frac{L}{H} m\pi\right) = \int_0^H \left(g(y) \sin\left(\frac{m\pi}{H} y\right)\right) dy$.
    - Finally: $\boxed{A_m = \frac{2}{H \sinh(-Lm\pi/H)} \int_0^H \left(g(y) \sin\left(\frac{m\pi}{H} y\right)\right) dy}$