SM315 Lecture Notes
Derivation of the Wave Equation
(138) 1ab*, 2

1. Configuration:

a. String vibrates only if it is stretched.
b. Ends are tied down to maintain a stretched nature.
c. Vibration is vertical displacement $y(x,t)$ or $u(x,t)$
d. $v(x,t)$ is horizontal displacement

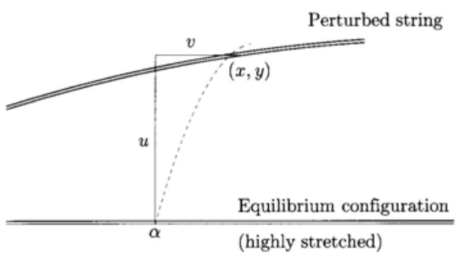

Figure 4.2.1 Vertical and horizontal displacements of a particle on a highly stretched string.

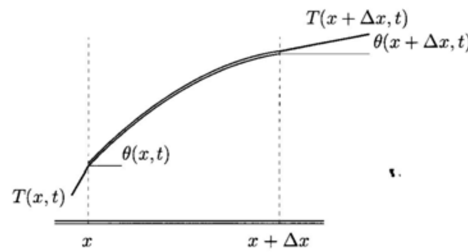

Figure 4.2.2 Stretching of a finite segment of string, illustrating the tensile forces.

2. Assumptions:

a. Slope of string is small – allows us to neglect horizontal displacement (i.e. motion is entirely vertical).
b. Mass density $\rho_0(x)$ is known.
c. All body forces (i.e. gravity) act in a vertical direction.
d. The only body force is gravity.
e. String is perfectly flexible and offers not resistance to bending.
f. String is perfectly elastic, which means magnitude of tensile force depends only on the local stretching of the string.
g. Tension $T(x,t)$ is a constant T_0
h. The body force due to gravity is much smaller that the force due to tension and can be neglected.
i. Assume angle θ is small
j. Assume string has uniform density.

3. Force Balance (along small length of string in Fig 4.2.2.):

a. $F = ma \rightarrow \rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin(\theta(x+\Delta x.t)) - T(x,t)\sin(\theta(x,t)) + \rho_0(x)\Delta x g$
   - Here $\rho_0(x)\Delta x$ is the expression for mass
   - $T_0 \sin(\theta(x+\Delta x),t) - T_0 \sin(\theta(x,t))$ is the approximate vertical force due to tension.

b. By assumption 2g equation becomes:

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T_0 \sin(\theta(x+\Delta x,t)) - T_0 \sin(\theta(x,t)) + \rho_0(x)\Delta x g$$

c. By assumption 2h equation becomes:

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T_0 \sin(\theta(x+\Delta x,t)) - T_0 \sin(\theta(x,t))$$

d. Now divide by Δx: $\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\sin(\theta(x+\Delta x,t)) - \sin(\theta(x,t))}{\Delta x}\right)$

e. Let $\Delta x \rightarrow 0$: $\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial}{\partial x}\sin(\theta(x,t))$

f. Now consider $\frac{\partial u}{\partial x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$ for small angles of $\theta \rightarrow \rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0(x)}\frac{\partial^2 u}{\partial x^2} \rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$

g. By assuming uniform density, c^2 is constant.

h. Classification: Hyperbolic

> **Recall 2nd Order Linear PDE w/2 Variables**
>
> $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$
>
> - **Parabolic**: $B^2 - 4AC = 0$
> - **Elliptic**: $B^2 - 4AC < 0$
> - **Hyperbolic**: $B^2 - 4AC > 0$

4. Boundary Conditions:

a. Ends of string are pinned: $u(0,t) = u(L,t) = 0$

b. Free end: $\frac{\partial u}{\partial x}(0,t) = 0$ or $\frac{\partial u}{\partial x}(L,t) = 0$

c. Forced Vibration: $u(0,t) = y(t)$ or $u(L,t) = y(t)$

5. Initial Condition:

a. Initial displacement or velocity profile of string i.e. $u(x,0) = y(t)$ or $\frac{d}{dt}u(x,0) = y(t)$ (plucked vs. struck string)