SM315 Lecture Notes
Klein-Gordon Equation with Fixed Ends
Homework: (Handout) 3

1. Klein Gordon Equation (The Dispersive Wave)

a. PDE: $a\dfrac{\partial^2 u}{\partial t^2} + cu = \omega^2 \dfrac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t) = u(L,t) = 0$

c. IC: $u(x,0) = f(x)$ & $\dfrac{\partial u}{\partial t}(x,0) = g(x)$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(x)T(t) \to u = \phi T$

a. Rewrite PDE as $au_{tt} + cu = \omega^2 u_{xx} \to a\phi T'' + c\phi T = \omega^2 \phi'' T$

b. Divide both sides by $\omega^2 \phi T \to \dfrac{a\phi T''}{\omega^2 \phi T} + \dfrac{c\phi T}{\omega^2 \phi T} = \omega^2 \dfrac{\phi'' T}{\omega^2 \phi T} \to \dfrac{a}{\omega^2}\dfrac{T''}{T} + \dfrac{c}{\omega^2} = \dfrac{\phi''}{\phi} = -\lambda$

- Note: Carry ω^2 with T to simplify approach to final solution
- Note: λ must be a constant and is called an eigenvalue
- Note: Used $-\lambda$, again simplify approach to final solution

3. Eigenvalue analysis using boundary conditions.

a. DE in space: $\dfrac{\phi''}{\phi} = -\lambda \to \phi'' = -\lambda\phi \to \phi'' + \lambda\phi = 0$

b. Eigenvalue analysis is exactly as that for the heat equation with $T = 0$ ends

- $\lambda < 0$ → trivial solution
- $\lambda = 0$ → trivial solution
- $\lambda < 0$ → $\boxed{\lambda_n = \left(\dfrac{n\pi}{L}\right)^2}$ and $\boxed{\phi_n = c_n \sin\left(\sqrt{\lambda_n}\, x\right)}$

4. DE in Time

a. DE in time:
$$\begin{cases} \dfrac{a}{\omega^2}\dfrac{T''}{T} + \dfrac{c}{\omega^2} = -\lambda_n \to aT'' + cT = -\lambda_n \omega^2 T \to \\ aT'' + \left(c + \lambda_n \omega^2\right)T = 0 \to T'' + \dfrac{c + \lambda_n \omega^2}{a} T = 0 \end{cases}$$

b. Let $\alpha_n^2 = \dfrac{c + \lambda_n \omega^2}{a} \to T'' + \alpha_n^2 T = 0$

c. Recall $T'' + \alpha_n^2 T = 0 \to T_n = A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)$

5. Put it all together to get $u(x,t)$

a. $u(x,t) = \phi(x)T(t) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n}\,x\right)\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right)$

b. Combine constants: $\sum_{n=1}^{\infty} \sin\left(\sqrt{\lambda_n}\,x\right)\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right)$

6. Apply Initial Condition

a. $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\sqrt{\lambda_n}\,x\right) = f(x) \rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin(\lambda_n x)dx$

b.
$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\sqrt{\lambda_n}\,x\right)\left(-A_n \sin.(\alpha_n t) + B_n \cos(\alpha_n t)\right) \rightarrow \\ \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \alpha_n B_n \sin\left(\sqrt{\lambda_n}\,x\right)\cos(\alpha_n t) = g(x) \rightarrow \\ \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \alpha_n B_n \sin\left(\sqrt{\lambda_n}\,x\right) = g(x) \rightarrow \\ \alpha_n B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\sqrt{\lambda_n}\,x\right)dx \rightarrow B_n = \frac{2}{\alpha_n L}\int_0^L g(x)\sin\left(\sqrt{\lambda_n}\,x\right)dx \end{cases}$$

7. Final Solution

a. Plug in values for λ_n and α_n

b. Recall α_n^2
$$\begin{cases} = \frac{c + \lambda_n \omega^2}{a} = \frac{c + \left(\frac{n\pi}{L}\right)^2 \omega^2}{a} = \frac{cL^2 + (n\pi\omega)^2}{aL^2} \rightarrow \\ \alpha_n = \sqrt{\frac{cL^2 + (n\pi\omega)^2}{aL^2}} = \frac{\sqrt{acL^2 + a(n\pi\omega)^2}}{aL} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{\sqrt{acL^2 + a(n\pi\omega)^2}}{aL}t\right) + B_n \sin\left(\frac{\sqrt{acL^2 + a(n\pi\omega)^2}}{aL}t\right)\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2aL}{\sqrt{acL^2 + a(n\pi\omega)^2}}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

8. Example

a.
$$\begin{array}{|l|}\hline PDE: \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{dx^2}, \quad 0 < x < \pi, \quad t > 0 \\ BC: \quad u(0,t) = u(\pi,t) = 0 \\ IC: \quad u(x,0) = x(\pi - x) \quad \& \quad \frac{\partial u}{\partial t}(x,0) = 0 \\ \hline \end{array}$$

b. This example implies that
$a = 1, \quad c = 1, \quad \omega^2 = 1, \quad L = \pi, \quad f(x) = x(\pi - x), \quad \& \quad g(x) = 0$

c. Therefore (substituting in general equation above):

d.
$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(A_n \cos\left(\frac{\sqrt{\pi^2 + (n\pi)^2}}{\pi} t \right) + B_n \sin\left(\frac{\sqrt{\pi^2 + (n\pi)^2}}{\pi} t \right)\right) \to \\ u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(A_n \cos\left(\sqrt{1+n^2}\, t\right) + B_n \sin\left(\sqrt{1+n^2}\, t\right)\right) \\ A_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)dx = \frac{4}{n^3 \pi}\left(1 - (-1)^n\right) \\ B_n = \left(1 + n^2\right)\int_0^{L} 0 \sin(nx)dx = 0 \end{cases}$$

e. Therefore: $\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^3}\left(1 - (-1)^n\right)\right]\cos\left(\sqrt{1+n^2}\, t\right)\sin(nx)}$

9. Maple Plot

10. Example

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t) = u(\pi, t) = 0$

c. IC: $u(x,0) = \begin{cases} x & \text{for} \quad 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for} \quad \frac{\pi}{2} < x < \pi \end{cases}$ and $\frac{\partial u}{\partial t}(x,0) = g(x) = 0$

d. General Solution applies with values $L = \pi$ and $c^2 = 1$ plugged in:

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(a_n \cos(nt) + b_n \sin(nt)\right) \\ a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x \sin(nx)dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)dx\right) \\ b_n = \frac{2}{nc\pi}\int_0^L g(x)\sin(nx)dx = 0 \end{cases}$$

e. Therefore: $\begin{cases} u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nt) \\ a_n = \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi}\sin(n\pi) = \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) \end{cases}$

11. Maple Demo – Modes of Vibration and Animation of Example.