SM315 Lecture Notes
Telegraph Equation with Fixed Ends
Homework: (Handout) 4

1. Telegraph Equation (Dispersive Waves w/Attenuation)

- PDE: $a\dfrac{\partial^2 u}{\partial t^2}+b\dfrac{\partial u}{\partial t}+cu=\omega^2\dfrac{\partial^2 u}{\partial x^2}$
- BC: $u(0,t)=u(L,t)=0$
- IC: $u(x,0)=f(x)$ & $\dfrac{\partial u}{\partial t}(x,0)=0$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t)=\phi(x)T(t)\rightarrow u=\phi T$

- Rewrite PDE as $au_{tt}+bu_t+cu=\omega^2 u_{xx}\rightarrow a\phi T''+b\phi T'+c\phi T=\omega^2\phi'' T$
- Divide both sides by

$$\omega^2\phi T\rightarrow \frac{a\phi T''}{\omega^2\phi T}+\frac{a\phi T'}{\omega^2\phi T}+\frac{c\phi T}{\omega^2\phi T}=\omega^2\frac{\phi'' T}{\omega^2\phi T}\rightarrow \frac{a}{\omega^2}\frac{T''}{T}+\frac{b}{\omega^2}\frac{T'}{T}+\frac{c}{\omega^2}=\frac{\phi''}{\phi}=-\lambda$$

  - Note: Carry ω^2 with T to simplify approach to final solution
  - Note: λ must be a constant and is called an eigenvalue
  - Note: Used $-\lambda$, again simplify approach to final solution

3. Eigenvalue analysis using boundary conditions.

- DE in space: $\dfrac{\phi''}{\phi}=-\lambda\rightarrow\phi''=-\lambda\phi\rightarrow\phi''+\lambda\phi=0$

- Eigenvalue analysis is exactly as that for the heat equation with $T=0$ ends
  - $\lambda<0$ → trivial solution
  - $\lambda=0$ → trivial solution
  - $\lambda<0$ → $\boxed{\lambda_n=\left(\dfrac{n\pi}{L}\right)^2}$ and $\boxed{\phi_n=c_n\sin\left(\sqrt{\lambda_n}\,x\right)=c_n\sin\left(\dfrac{n\pi x}{L}\right)}$

4. DE in Time

- DE in time: $\begin{cases} \frac{a}{\omega^2}\frac{T''}{T} + \frac{b}{\omega^2}\frac{T'}{T} + \frac{c}{\omega^2} = -\lambda_n \rightarrow aT'' + bT' + cT = -\lambda_n \omega^2 T \rightarrow \\ aT'' + bT' + \left(c + \lambda_n \omega^2\right)T = 0 \rightarrow T'' + \frac{b}{a}T' + \frac{c + \lambda_n \omega^2}{a}T = 0 \end{cases}$

- Let $\alpha_n^2 = \frac{c + \lambda_n \omega^2}{a}$ *and* $\beta = \frac{b}{a} \rightarrow T'' + \beta T' + \alpha_n^2 T = 0$
    - Note: no *n* subscript needed for β

- Solve using notation used in DE: $\left(D^2 + \beta D + \alpha_n\right)T = 0 \rightarrow D = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha_n^2}}{2}$

- Assuming $\beta^2 < 4\alpha_n^2 \rightarrow D - \frac{\beta}{2} \pm \frac{\sqrt{4\alpha_n^2 - \beta^2}}{2} i$

- Therefore

$$T'' + \beta T' + \alpha_n^2 T = 0 \rightarrow T_n = A_n e^{\frac{-\beta}{2}t} \sin\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) + B_n e^{\frac{-\beta}{2}t} \cos\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right)$$

5. Apply Initial Conditions

- Apply the zero-initial condition $\frac{\partial^2 u}{\partial t^2}(x,0) = 0$. Unfortunately, this leaves us with the messy proposition of determining T_n'

- $$\begin{cases} T_n'(t) = A_n\left(\frac{-\beta}{2}\right)e^{\frac{-\beta}{2}t} \sin\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) + A_n\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}\right)e^{\frac{-\beta}{2}t} \cos\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) \\ \qquad + B_n\left(\frac{-\beta}{2}\right)e^{\frac{-\beta}{2}t} \cos\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) - B_n\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}\right)e^{\frac{-\beta}{2}t} \sin\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) = 0 \rightarrow \\ T_n(0) = A_n\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}\right) - B_n\left(\frac{\beta}{2}\right) = 0 \rightarrow B_n = \left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{\beta}\right)A_n \end{cases}$$

- Unfortunately, we did not completely eliminate A_n or B_n but the result is useful and we can rewrite $T_n(t) = A_n e^{\frac{-\beta}{2}t}\left(\sin\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right) + \left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{\beta}\right)\cos\left(\frac{\sqrt{4\alpha_n^2 - \beta^2}}{2}t\right)\right)$

- Recall from Physics or ODE that an expression: $A\sin(\omega t)+B\cos(\omega t)$ can be rewritten as $C\sin(\omega t+\phi)$ where $C=\sqrt{A^2+B^2}$ and $\phi=\arctan\left(\frac{B}{A}\right)$.

- Therefore: $T_n(t)=A_n e^{\frac{-\beta}{2}t}\left(\frac{2\alpha_n}{\sqrt{4\alpha_n^2-\beta^2}}\sin\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}t+\arctan\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}\right)\right)\right)$

6. Put it all together to get $u(x,t)$

$$u(x,t)=\phi(x)T(t)=\sum_{n=1}^{\infty}c_n\sin\left(\sqrt{\lambda_n}\,x\right)e^{\frac{-\beta}{2}t}\left(A_n\left(\sin\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}t\right)+\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{\beta}\right)\cos\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}t\right)\right)\right)$$

And constants:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\sqrt{\lambda_n}\,x\right)e^{\frac{-\beta}{2}t}\left(\sin\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}t\right)+\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{\beta}\right)\cos\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{2}t\right)\right)}$$

7. Apply 2nd Initial Condition

- $u(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\sqrt{\lambda_n}\,x\right)\left(\frac{\sqrt{4\alpha_n^2-\beta^2}}{\beta}\right)=f(x)\rightarrow$

  $$\boxed{A_n=\frac{2}{L}\frac{\beta}{\sqrt{4\alpha_n^2-\beta^2}}\int_0^L f(x)\sin(\lambda_n x)dx}$$

8. Final Solution

- Plug in values for λ_n and α_n → recall $\lambda_n=\left(\frac{n\pi}{L}\right)^2$
- $\alpha_n^2=\frac{c+\lambda_n\omega^2}{a}=\frac{c+\left(\frac{n\pi}{L}\right)^2\omega^2}{a}=\frac{cL^2+(n\pi\omega)^2}{aL^2}$ and $\beta=\frac{b}{a}$
- Granted the final form of the solution is mess, but it cleans up nicely if I use numbers

9. Example

- $$\begin{array}{l} PDE: \ \dfrac{\partial^2 u}{\partial t^2} + .2\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{dx^2}, \quad 0 < x < \pi, \quad t > 0 \\ BC: \quad u(0,t) = u(\pi,t) = 0 \\ IC: \quad u(x,0) = x(\pi - x) \ \ \& \ \ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{array}$$

- Separation of variables yields: $\begin{cases} \phi'' + \lambda\phi = 0 \\ T'' + .2T' + \lambda T = 0 \end{cases}$

- BC analysis yields $\boxed{\lambda_n = n^2}$ and $\boxed{\phi_n = c_n \sin(nx)}$

- Solving equation in T:
$$\begin{cases} T'' + .2T' + \lambda T = 0 \rightarrow (D^2 + .2D + n^2)T = 0 \rightarrow \\ D = \dfrac{-.2 + \sqrt{.04 - 4n^2}}{2} = \dfrac{-.2 + \sqrt{4n^2 - .04}i}{2} = -.1 + \sqrt{n^2 - .01}i \rightarrow \\ T_n(t) = A_n e^{-.1t} \sin\left(\sqrt{n^2 - .01}t\right) + B_n e^{-.1t} \cos\left(\sqrt{n^2 - .01}t\right) \end{cases}$$

- Find $T_n'(t)$ so that we can evaluate the zero initial condition:
$$\begin{cases} T_n'(t) = -.1A_n e^{-.1t} \sin\left(\sqrt{n^2 - .01}t\right) + \sqrt{n^2 - .01}A_n e^{-.1t} \cos\left(\sqrt{n^2 - .01}t\right) \\ \qquad\qquad -.1B_n e^{-.1t} \cos\left(\sqrt{n^2 - .01}t\right) - \sqrt{n^2 - .01}A_n e^{-.1t} \sin\left(\sqrt{n^2 - .01}t\right) \rightarrow \\ T_n'(0) = \sqrt{n^2 - .01}A_n - .1B_n = 0 \rightarrow B_n = 10\sqrt{n^2 - .01}A_n \rightarrow \\ T_n(t) = A_n e^{-.1t}\left(\sin\left(\sqrt{n^2 - .01}t\right) + 10\sqrt{n^2 - .01}\cos\left(\sqrt{n^2 - .01}t\right)\right) \end{cases}$$

- Put it all together
$$u(x.t) = \sum_{n=0}^{\infty} A_n e^{-.1t}\left(\sin\left(\sqrt{n^2 - .01}t\right) + 10\sqrt{n^2 - .01}\cos\left(\sqrt{n^2 - .01}t\right)\right)\sin(nx)$$

- Apply second initial condition:
$$\begin{cases} u(x.0) = \sum_{n=0}^{\infty} A_n\left(10\sqrt{n^2 - .01}\right)\sin(nx) = x(\pi - x) \rightarrow \\ A_n = \dfrac{1}{10\sqrt{n^2 - .01}} \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x(\pi - x)\sin(nx)dx = \dfrac{1}{5\pi\sqrt{n^2 - .01}}\left(\dfrac{2 - 2(-1)^n}{n^3}\right) \rightarrow \\ A_n = \dfrac{2 - 2(-1)^n}{5n^3\pi\sqrt{n^2 - .01}} \end{cases}$$

- The solution is complete. We leave it to maple to plot.