SM315 Lecture Notes
Vibrating String with Fixed Ends and a Forcing Function
(Eigenvalue Expansion Method)
Homework: (Handout) 5

1. Heat Equation

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t)$

b. BC: $u(0,t) = u(L,t) = 0$

c. IC: $u(x,0) = f(x)$ & $\frac{\partial u}{\partial t}(x,0) = g(x)$

2. Assumptions Based on Experience

a. Solution will have form $u(x,t) = \sum_{n=0}^{\infty} T_n(t)\phi_n(x)$

b. Based on boundary conditions, $\phi_n(x)$ will come from the DE $\phi_n'' + \lambda\phi_n = 0$ where
   - $\phi_n = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$
   - $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

c. Therefore solution has form $\sum_{n=0}^{\infty} c_n T_n(t)\sin\left(\frac{n\pi x}{L}\right) \rightarrow \sum_{n=0}^{\infty} T_n(t)\sin\left(\frac{n\pi x}{L}\right)$
   - (i.e. combining constant c_n with $T_n(t)$

d. Initial conditions become:
   - $u(x,0) = \sum_{n=0}^{\infty} T_n(0)\sin\left(\frac{n\pi x}{L}\right) = f(x) \rightarrow T_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$
   - $\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} T_n{}'(0)\sin\left(\frac{n\pi x}{L}\right) = g(x) \rightarrow T_n'(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

3. Plug Assumed Solution Back into ODE.

a.
$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\left[\sum_{n=0}^{\infty} T_n(t)\sin\left(\dfrac{n\pi x}{L}\right)\right] = \omega^2 \dfrac{\partial^2}{\partial x^2}\left[\sum_{n=0}^{\infty} T_n(t)\sin\left(\dfrac{n\pi x}{L}\right)\right] + Q(x,t) \rightarrow \\ \sum_{n=0}^{\infty} T_n''(t)\sin\left(\dfrac{n\pi x}{L}\right) = -\omega^2 \sum_{n=0}^{\infty}\left(\dfrac{n\pi}{L}\right)^2 T_n(t)\sin\left(\dfrac{n\pi x}{L}\right) + Q(x,t) \rightarrow \\ \sum_{n=0}^{\infty}\left(T_n''(t) + \left(\dfrac{\omega n\pi}{L}\right)^2 T_n(t)\right)\sin\left(\dfrac{n\pi x}{L}\right) = Q(x,t) \end{cases}$$

b. Now apply a Fourier Series like Step (works because of the orthogonality of sines→

$$\begin{cases} T_n''(t) + \left(\dfrac{\omega n\pi}{L}\right)^2 T_n(t) = \dfrac{2}{L}\int_0^L Q(x,t)\sin\left(\dfrac{n\pi x}{L}\right)dx = q(t) \rightarrow \\ \left(D^2 + \left(\dfrac{\omega n\pi}{L}\right)^2\right)T_n = q(t) \end{cases}$$

c. The question now it … can I solve this ODE.

- I have 2 initial conditions from step 2d:
  1. $T_n(0) = \dfrac{2}{L}\int_0^L f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$
  2. $T_n'(0) = \dfrac{2}{L}\int_0^L g(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$

- I have some techniques to solve non homogenious ODEs with constant coefficients:
  1. Undetermined Coefficients
  2. Annihilators
  3. Variation of Parameters
  4. Laplace Transforms

4. Example

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + tx \rightarrow \omega^2 = 1 \;\; and \;\; Q(x,t) = tx$

b. BC: $u(0,t) = u(\pi,t) = 0 \rightarrow L = \pi$

c. IC: $\begin{cases} u(x,0) = f(x) = \sin(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) = 5\sin(2x) \end{cases}$

d. From 2b: $\phi_n = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2$ which implies $\phi_n = c_n \sin(nx)$

e. From 2d initial conditions become:

- $T_n(0) = \frac{2}{\pi}\int_0^\pi \sin(x)\sin(nx)dx \rightarrow \begin{cases} T_1(0) = 1 \\ T_n(0) = 0 \quad for \;\; all \;\; n \neq 1 \end{cases}$
- $T_n'(0) = \frac{2}{\pi}\int_0^\pi 5\sin(2x)\sin(nx)dx \rightarrow \begin{cases} T_2(0) = 5 \\ T_n(0) = 0 \quad for \;\; all \;\; n \neq 5 \end{cases}$

f. From 3b ODE for $T(t)$:

- $\begin{cases} T_n''(t) + n^2 T_n(t) = \frac{2}{\pi}\int_0^\pi tx\sin(nx)dx = \frac{2t}{\pi}\int_0^\pi x\sin(nx)dx \rightarrow \\ \left(D^2 + n^2\right)T_n = \frac{2}{n}(-1)^{n+1} t \end{cases}$

g. Use Annihilators to solve:

- $D^2\left(D^2 + n^2\right)T_n = D^2\left(\frac{2}{n}(-1)^{n+1} t\right) = 0$
- $T_n = c_1 t + c_2 + c_3 \sin(nt) + c_4 \cos(nt)$

  a. Plug particular solution $c_1 t + c_2$ into the original DE to get c_1 and c_2:

  - $\begin{cases} \left(D^2 + n^2\right)\left(c_1 t + c_2\right) = n^2\left(c_1 t + c_2\right) = \frac{2}{n}(-1)^{n+1} t \rightarrow \\ c_2 = 0 \;\; and \;\; c_1 = \frac{2}{n^3}(-1)^{n+1} \rightarrow \\ T_n(t) = \left(\frac{2}{n^3}(-1)^{n+1}\right)t + c_3\sin(nt) + c_4\cos(nt) \end{cases}$

h. Apply initial conditions to get c_3 and c_4:

- $T_n(0) = c_{4n} = \begin{cases} 1 & for \quad n = 1 \\ 0 & for \quad n \neq 0 \end{cases}$

- $\begin{cases} T_n'(t) = \dfrac{2}{n^3}(-1)^{n+1} + c_3 n\cos(nt) + c_4 n\sin(nt) \\ T_n'(0) = \dfrac{2}{n^3}(-1)^{n+1} + c_{3n} n = \begin{cases} 2 & for \quad n = 2 \\ 0 & for \quad n \neq 2 \end{cases} \rightarrow \\ c_{3n} = \begin{cases} 42/16 & for \quad n = 2 \\ (2/n^4)(-1)^n & for \quad n \neq 2 \end{cases} \end{cases}$

i. Finally:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x) =$$
$$(2t + \cos(t) - 2\sin(t))\sin(x) + \left(-\frac{1}{4}t + \frac{21}{8}\sin(2t)\right)\sin(2x) + \sum_{n=3}^{\infty}\left(\left(\frac{2}{n^3}(-1)^{n+1}\right)t + \frac{2}{n^4}(-1)^n\sin(nt)\right)\sin(nx)$$

j. Check:

- $u(x,0) = (2-0)\sin(x) = 2\sin(x)$ (all other terms go to 0)

- $\dfrac{\partial u}{\partial t}(x,t) = \begin{cases} (2 - \sin(t) - 2\cos(t))\sin(x) \\ + \left(-\dfrac{1}{4} + \dfrac{21}{4}\cos(2t)\right)\sin(2x) \\ + \displaystyle\sum_{n=3}^{\infty}\left(\left(\frac{2}{n^3}(-1)^{n+1}\right) + \frac{2}{n^3}(-1)^n\cos(nt)\right)\sin(nx) \end{cases} \rightarrow$

- $\dfrac{\partial u}{\partial t}(x,0) = \begin{cases} (2-2)\sin(x) + \left(-\dfrac{1}{4} + \dfrac{21}{4}\right)\sin(2x) \\ + \displaystyle\sum_{n=3}^{\infty}\left(\left(\frac{2}{n^3}(-1)^{n+1}\right) - \frac{2}{n^3}(-1)^{n+1}\right)\sin(nx) \end{cases} \rightarrow$

- $\dfrac{\partial u}{\partial t}(x,0) = \sin(2x)$