SM315 Lecture Notes
Vibrating Rectangular Membrane
Homework: (279) 4a, 5

1. Wave Equation in 2D Rectangular Coordinates

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \omega^2 \nabla^2 u = \omega^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

b. BC: $\begin{cases} u(0, y, t) = u(L, y, t) = 0 \\ u(x, 0, t) = u(x, H, t) = 0 \end{cases}$

c. IC: $\begin{cases} u(x, y, 0) = f(x, y) \\ \frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \end{cases}$

2. Separate the Variables

a. $u(x, y, t) = X(x)Y(y)T(t)$ or in shorthand $u = XYT$

b. PDE becomes: $XYT'' = \omega^2 X''YT + \omega^2 XY''T$

c. Now divide by $\omega^2 XYT \rightarrow$ PDE becomes: $\frac{1}{\omega^2} \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$

d. Note that the right hand side is a function of t only, and the left hand side is a function of x and y only. The only way for this equality to exist is for both sides to be equal to the same constant $-\lambda$, i.e.: $\frac{1}{\omega^2} \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$

e. This generates our first ODE for T: $T'' + \lambda \omega^2 T = 0$

f. In order to evaluate the eigenvalue, boundary conditions must be applied. But X and Y are not yet separated.

g. So we proceed in this manner:

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \lambda$$

h. X and Y are now separated, but the expression requires the introduction of a second eigenvalue μ, i.e.: $\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$.

i. This allows us to define two additional ODES for X and Y. The three ODEs are:

$$\begin{cases} T'' + \lambda \omega^2 T = 0 \\ X'' + \mu X = 0 \\ Y'' + (\lambda - \mu) Y = 0 \end{cases}$$

3. Use the BC's in *x* to Evaluate the Eigenvalue μ

a. There is nothing new here (note boundary conditions imply that $\mu > 0$ for a non-trivial solution). Therefore:

$$\begin{cases} X'' + \mu X = 0 \to X = c_1 \sin\left(\sqrt{\mu}x\right) + c_2 \cos\left(\sqrt{\mu}x\right) \\ X(0) = c_2 = 0 \to X = c_1 \sin\left(\sqrt{\mu}x\right) \\ X(L) = c_1 \sin\left(\sqrt{\mu}L\right) = 0 \to \sqrt{\mu}L = n\pi \to \\ \mu_n = \left(\frac{n\pi}{L}\right)^2 \quad and \quad X_n = c_n \sin\left(\frac{n\pi x}{L}\right) \end{cases}$$

4. Use the BC's in y to Evaluate the Eigenvalue λ

a. Note much new here, just a twist (note boundary conditions imply that $\lambda + \mu_n$ for a non-trivial solution). Therefore:

$$\begin{cases} Y'' + (\lambda - \mu_n)Y = 0 \to \\ Y = b_1 \sin\left(\sqrt{\lambda - \mu_n}\, y\right) + b_2 \cos\left(\sqrt{\lambda - \mu_n}\, y\right) \\ Y(0) = b_2 = 0 \to Y = b_1 \sin\left(\sqrt{\lambda - \mu_n}\, y\right) \\ Y(H) = b_1 \sin\left(\sqrt{\lambda - \mu_n}\, H\right) = 0 \to \sqrt{\lambda - \mu_n}\, H = m\pi \to \\ \lambda = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \to \sqrt{\lambda_{mn} - \mu_n} = \frac{m\pi}{H} \\ Y_m = b_m \sin\left(\frac{m\pi y}{H}\right) \end{cases}$$

5. Use Find the General Solution for the ODE in Time

a. $$\begin{cases} T'' + \lambda_{mn}\omega^2 T = 0 \to d_1 \sin\left(\omega\sqrt{\lambda_{mn}}\, t\right) + d_2 \cos\left(\omega\sqrt{\lambda_{mn}}\, t\right) \to \\ T = d_1 \sin\left(\omega\sqrt{\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\, t\right) + d_2 \cos\left(\omega\sqrt{\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\, t\right) \end{cases}$$

b. Note we had to add the use of an *m* to keep the indices independent from one-another.

6. Put it all Together to Find *u(x,y,t)*

a. $$\begin{cases} u_{mn}(x,y,t) = X_n(x)Y_m(y)T_{mn}(t) \to \\ u_{mn}(x,y,t) = c_n \sin\left(\frac{n\pi x}{L}\right) b_m \sin\left(\frac{m\pi y}{H}\right)\left[d_1 \sin\left(\omega\sqrt{\lambda_{mn}}\, t\right) + d_2 \cos\left(\omega\sqrt{\lambda_{mn}}\, t\right)\right] \to \\ u_{mn}(x,y,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[b_{mn} \sin\left(\omega\sqrt{\lambda_{mn}}\, t\right) + a_{mn} \cos\left(\omega\sqrt{\lambda_{mn}}\, t\right)\right] \end{cases}$$

b. To get the expression for *u*, we must sum over the indices *m* and *n*. This requires a double summation expression:

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[b_{mn}\sin\left(\omega\sqrt{\lambda_{mn}}\,t\right)+a_{mn}\cos\left(\omega\sqrt{\lambda_{mn}}\,t\right)\right]$$

- Note: Whether you sum over *m* first or *n* first is irrelevant.

7. Using Initial Conditions to Find a_{mn}

a. $u(x,y,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)=f(x,y)$

b. Multiply both sides by $\sin\left(\frac{k\pi x}{L}\right)$ and integrate:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\left[\int_{x=0}^{L}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right]\sin\left(\frac{m\pi y}{H}\right)=\int_{x=0}^{L}f(x,y)\sin\left(\frac{k\pi x}{L}\right)dx$$

c. Note that the integral in the brackets only has a non-zero value if $n=m$, i.e.:

$$\int_{0}^{L}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx=\begin{cases}0 & if \quad k\neq n\\ \frac{L}{2} & if \quad k=n\end{cases}.$$

d. Hence we can eliminate the inside summation and rewrite the equation.

$$\begin{cases}\sum_{m=1}^{\infty}a_{mn}\left[\frac{L}{2}\right]\sin\left(\frac{m\pi y}{H}\right)=\int_{0}^{L}f(x,y)\sin\left(\frac{n\pi x}{L}\right)dx\rightarrow\\ \sum_{m=1}^{\infty}a_{mn}\sin\left(\frac{m\pi y}{H}\right)=\frac{2}{L}\int_{0}^{L}f(x,y)\sin\left(\frac{n\pi x}{L}\right)dx\end{cases}$$

e. Multiply both sides by $\sin\left(\frac{k\pi y}{H}\right)$ and integrate:

$$\sum_{m=1}^{\infty}a_{mn}\left[\int_{o}^{H}\sin\left(\frac{k\pi y}{H}\right)\sin\left(\frac{m\pi y}{H}\right)dy\right]=\frac{2}{L}\int_{0}^{H}\left[\int_{0}^{L}f(x,y)\sin\left(\frac{k\pi y}{H}\right)\sin\left(\frac{m\pi x}{L}\right)dx\right]dy$$

f. Again note that the integral in the brackets only has a non-zero value if $m=k$, i.e.:

$$\int_{0}^{H}\sin\left(\frac{k\pi x}{H}\right)\sin\left(\frac{m\pi x}{H}\right)dx=\begin{cases}0 & if \quad k\neq m\\ \frac{H}{2} & if \quad k=m\end{cases}$$

g. Hence we can eliminate the inside summation and rewrite the equation.

$$\begin{cases}a_{mn}\left[\frac{H}{2}\right]=\frac{2}{L}\int_{0}^{H}\left[\int_{0}^{L}f(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dx\right]dy\rightarrow\\ a_{mn}=\frac{4}{LH}\int_{0}^{H}\int_{0}^{L}f(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dxdy\end{cases}$$

8. Using Initial Conditions to Find b_{mn}

a. First find an expression for $\frac{\partial u}{\partial t}$:

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[\omega\sqrt{\lambda_{mn}}b_{mn}\cos\left(\omega\sqrt{\lambda_{mn}}t\right) - \omega\sqrt{\lambda_{mn}}a_{mn}\sin\left(\omega\sqrt{\lambda_{mn}}t\right)\right] \rightarrow \\ \frac{\partial u}{\partial t}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega\sqrt{\lambda_{mn}}b_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) = g(x,y) \end{cases}$$

b. Let $c_{mn} = \omega\sqrt{\lambda_{mn}}b_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

c. Using steps similar to those above, we find:

$$\begin{cases} c_{mn} = \frac{4}{LH}\int_0^H\int_0^L g(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy \rightarrow \\ b_{mn} = \frac{1}{\omega\sqrt{\lambda_{mn}}}\frac{4}{LH}\int_0^H\int_0^L g(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy \end{cases}$$

9. In Summary … Our Solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[b_{mn}\sin\left(\omega\sqrt{\lambda_{mn}}t\right) + a_{mn}\cos\left(\omega\sqrt{\lambda_{mn}}t\right)\right]$$

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2$$

$$a_{mn} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

$$b_{mn} = \frac{1}{\omega\sqrt{\lambda_{mn}}}\frac{4}{LH}\int_0^H\int_0^L g(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

10. Example

a. PDE: $\frac{\partial^2 u}{\partial t^2} = 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ i.e. $\omega = \sqrt{2}$

b. BC: $\begin{cases} u(0, y, t) = u(\pi, y, t) = 0 \\ u(x, 0, t) = u(x, 2\pi, t) = 0 \end{cases}$ i.e. $L = \pi,\ H = 2\pi$

c. IC: $\begin{cases} u(x, y, 0) = f(x, y) = x^3(\pi - x)y(\pi - y)^4 \\ \frac{\partial u}{\partial t}(x, y, 0) = g(x, y) = 0 \end{cases}$

d. Plug and chug …

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin(nx)\sin\left(\frac{my}{2}\right)\cos\left(\sqrt{2\lambda_{mn}}\,t\right)$$

$$\lambda_{mn} = \left(\frac{1}{2}m\right)^2 + (n)^2$$

$$a_{mn} = \frac{2}{\pi^2}\int_0^{2\pi}\int_0^{\pi} x^3(\pi - x)y(\pi - y)^4 \sin(nx)\sin\left(\frac{my}{2}\right)dxdy$$

$$b_{mn} = 0$$

e. Maple Demo