SM315 Lecture Notes
Vibrating Circular Membrane
Homework: (TBD)

1. Wave Equation in 2D Polar Coordinates

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \omega^2 \nabla^2 u = \omega^2 \left(\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$ where $\begin{cases} 0 \le r \le a \\ -\pi \le \theta \le \pi \\ t \ge 0 \end{cases}$

b. BC: $\begin{cases} u(a,\theta,t) = 0 \\ |u(0,\theta,t)| < \infty \end{cases}$ and $\begin{cases} u(r,-\pi,t) = u(r,\pi,t) \\ \frac{\partial u}{\partial \theta}(r,-\pi,t) = \frac{\partial u}{\partial \theta}(r,\pi,t) \end{cases}$

- $u(a,\theta,t) = 0$ implies that outside edge of circle is pinned down (i.e. like a drum head).
- $|u(0,\theta,t)| < \infty$ implies that the displacement must be finite.

c. IC: $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$

- BC's in theta are due to the periodic nature of the solution

2. Separate the Variables

a. Expand the PDE by applying product rule to $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ term, i.e.:

$$\frac{\partial^2 u}{\partial t^2} = \omega^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

b. $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ or in shorthand $u = R\Theta T$

c. PDE becomes: $R\Theta T'' = \omega^2 R''\Theta T + \frac{\omega^2}{r}R'\Theta T + \omega^2 R\Theta'' T$

d. Now divide by $\omega^2 R\Theta T \rightarrow$ PDE becomes:

$$\frac{1}{\omega^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{\Theta''}{\Theta}$$

e. Note that the right hand side is a function of t only, and the left hand side is a function of r and θ only. The only way for this equality to exist is for both sides to be equal to the same constant $-\lambda$, i.e.:

$$\frac{1}{\omega^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{\Theta''}{r^2\Theta} = -\lambda =$$

f. This generates the familiar ODE for T: $T'' + \lambda\omega^2 T = 0$

g. In order to evaluate the eigenvalue, boundary conditions must be applied. But r and θ are not yet separated.

h. In order to separate r from θ we must multiply $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{\Theta''}{r^2\Theta} = -\lambda$ by r^2, i.e.

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = -\lambda$$

i. So we proceed in this manner separating r from θ :

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

- The reason for using μ vice $-\mu$ is evident in the next step

j. This generates the ODE for θ: $\Theta'' + \mu\Theta = 0$

k. And finally the ODE for R: $r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0$

l. In Summary, the three ODEs are:

$$\begin{cases} T'' + \lambda\omega^2 T = 0 \\ \Theta'' + \mu\Theta = 0 \\ r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0 \end{cases}$$

3. Use the BC's in θ to Evaluate the Eigenvalue μ

a. We have solved this boundary volume problem before when we solved the problem in a circular ring (See Table 2.4.1 on Page 69):

- $\mu_m = \left(\frac{m\pi}{L}\right)^2$ where $m = 0,1,2...$. We consider that $L = \pi$ since $-\pi \le \theta \le \pi$ and therefore $\mu_m = m^2$
- Thus $\Theta'' + m^2\Theta = 0 \to \Theta = A_m \sin(m\theta) + B_m \cos(m\theta)$
- At this point we can do nothing more with the θ BC

4. Use the BC's in r to Evaluate the Eigenvalue λ

a. The differential equation in r, although unfamiliar to us, is actually a very well known equation (after some manipulation)

b. We let
$$\begin{cases} z = \sqrt{\lambda} r \to r = z/\sqrt{\lambda} \to \\ \frac{dR}{dr} = \frac{dR}{dz}\frac{dz}{dr} = \sqrt{\lambda}\frac{dR}{dz} \to \\ \frac{d^2R}{dr^2} = \frac{d}{dr}\frac{dR}{dr} = \frac{d}{dr}\left[\sqrt{\lambda}\frac{dR}{dz}\right] = \lambda\frac{d^2R}{dz^2} \end{cases}$$

c. therefore:
$$\begin{cases} r^2\frac{d^2}{dr^2}R(r) + r\frac{d}{dr}R(r) + \left(\lambda r^2 - m^2\right)R(r) = 0 \to \\ \frac{z^2}{\lambda}\frac{d^2}{dz^2}R(z(r)) + \frac{z}{\sqrt{\lambda}}\frac{d}{dz}R(z(r)) + \left(z^2 - m^2\right)R(z(r)) = 0 \end{cases}$$

d. Now apply chain rule:
$$\begin{cases} \frac{d}{dz}R(z(r)) = \frac{dR}{dz}\frac{dz}{dr} = \frac{dR}{dz}\frac{d}{dr}\left(\sqrt{\lambda}r\right) = \sqrt{\lambda}\frac{dR}{dz} \\ \frac{d^2z}{dr^2} = \frac{d}{dr}\left(\sqrt{\lambda}\frac{dR}{dz}\right) = \sqrt{\lambda}\frac{d^2R}{dz^2}\frac{dz}{dr} = \sqrt{\lambda}\frac{d^2R}{dz^2}\frac{d}{dr}\left(\sqrt{\lambda}r\right) = \lambda\frac{d^2R}{dz^2} \end{cases}$$

e. Substituting this in b yields:
$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + \left(z^2 - m^2\right)R = 0$$

f. This is known as **Bessel's Equation of Order m**

> *Bessel's Differential Equation* which arises in numerous problems, especially in polar and cylindrical coordinates. **Bessel's Differential Equation** is defined as:
>
> $$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0$$
>
> where n is a non-negative real number. The solutions of this equation are called **Bessel Functions** of order n. Although the order n can be any real number, the scope of this section is limited to *non-negative integers*, i.e., $n = 0,1,2,3,\cdots$, unless specified otherwise.

5. Notes on Bessel

a. Bessel worked out a method of mathematical analysis involving what is now known as the Bessel function. He introduced this in 1817 in his study of a problem of Kepler of determining the motion of three bodies moving under mutual gravitation.

b. Bessel functions appear as coefficients in the series expansion of the indirect perturbation of a planet, that is the motion caused by the motion of the Sun caused by the perturbing body. In 1824 he developed Bessel functions more fully in a study of planetary perturbations and published a treatise on them in Berlin. It was not the first time that special cases of the functions had appeared, Jacob Bernoulli, Daniel Bernoulli, Euler and Lagrange having studied special cases of them earlier. In fact it was probably Lagrange's work on elliptical orbits that first suggested to Bessel to work on the Bessel functions.