SM315 Lecture Notes
Using Bessel Functions to Solve Wave Equation on a Circular Disk with Radial Symmetry
Homework: (315) 1,10

1. Recall: Wave Equation in 2D Polar Coordinates

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \omega^2 \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$ where $\begin{cases} 0 \le r \le a \\ -\pi \le \theta \le \pi \\ t \ge 0 \end{cases}$

b. BC: $\begin{cases} u(a,\theta,t) = 0 \\ |u(0,\theta,t)| < \infty \end{cases}$ and $\begin{cases} u(r,-\pi,t) = u(r,\pi,t) \\ \frac{\partial u}{\partial \theta}(r,-\pi,t) = \frac{\partial u}{\partial \theta}(r,\pi,t) \end{cases}$

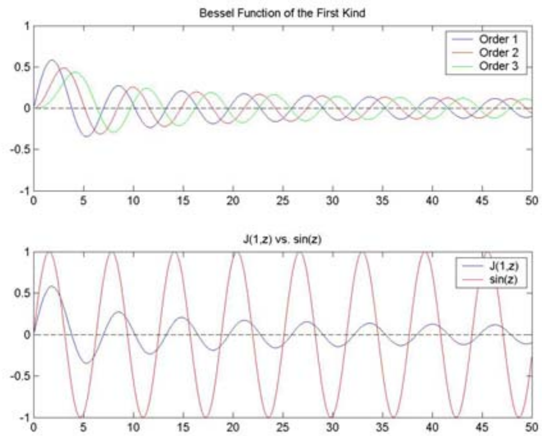
c. IC: $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,t)\theta \end{cases}$

d. Separation of variables produced the DE's: $\begin{cases} T'' + \lambda\omega^2 T = 0 \\ \Theta'' + \mu\Theta = 0 \\ r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0 \end{cases}$

e. After scaling the DE be letting $z = \sqrt{\lambda} r$ we generated $z^2 \frac{d^2 R}{dz^2} + z \frac{dR}{dz} + (z^2 - m^2)R = 0$ which is the well known ***Bessel's Equation of Order m***

f. The solution to this DE was developed by Bessel and is $R = c_1 J_m(z) + c_2 Y_m(z)$
   - $J_m(z)$ is called ***Bessel's function of the first kind of order m***.
   - $Y_m(z)$ is called ***Bessel's function of the second kind of order m***.

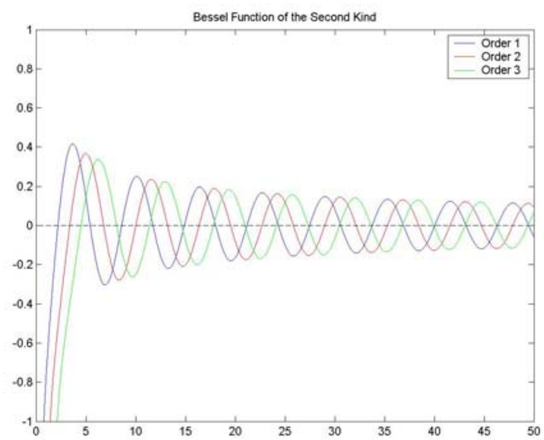
2. Observed Properties of the Bessel function of the First Kind $J_m(z)$



a. $J_m(0) = 0$
b. There are in infinite number or zeros (*but there is no closed form to express what they are*).
c. Looks like sin wave with infinite number of zero values
d. Has decaying amplitude.
e. As n increases the graph flattens out for a distance from zero. That distance increase with n.
f. Amplitude decreases as n increases.
g. Zero values seem to interlace, but not repeat for different values of n.
h. And a final property that is not observed from graphs … ***orthogonality***, i.e. For fixed order m, Bessel Functions are orthogonal with weight r:

$$\int_0^a J_m\left(\sqrt{\lambda_p}\, r\right) J_m\left(\sqrt{\lambda_q}\, r\right) r\, dr = 0 \quad for \quad p \neq q$$

3. Observed Properties of the Bessel function of the Second Kind $Y(n,z)$



a. $Y(n,0) \to -\infty$
b. There are in infinite number or zeros (*but there is no closed form to express what they are*).
c. Has decaying amplitude.
d. As n increases the graph "takes" longer to "hit" first zero value.
e. Amplitude decreases as n increases.
f. Zero values seem to interlace, but not repeat for different values of n.

4. Finding R(r) using Radial Symmetry

a. Radial Symmetry means that the wave function is independent of θ. This simplifies the PDE.

PDE: $\dfrac{\partial^2 u}{\partial t^2} = \omega^2\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)\right)$ where $\begin{cases} 0 \le r \le a \\ t \ge 0 \end{cases}$

BC: $\begin{cases} u(a,t) = 0 \\ |u(0,t)| < \infty \end{cases}$

IC: $\begin{cases} u(r,0) = f(r) \\ \dfrac{\partial u}{\partial t}(r,0) = g(r) \end{cases}$

b. Separation of variables produced the DE's: $\begin{cases} T'' + \lambda\omega^2 T = 0 \\ r^2 R'' + rR' + \lambda r^2 R = 0 \end{cases}$

c. After scaling the DE be letting $z = \sqrt{\lambda}r$ we generated $z^2\dfrac{d^2R}{dz^2} + z\dfrac{dR}{dz} + z^2 R = 0$

which is a known ***Bessel's Equation of Order m=0***

d. The solution of this equation is:

$$R(z) = c_1 J_0(z) + c_2 Y_0(z) \to R(r) = c_1 J_0\left(\sqrt{\lambda}r\right) + c_2 Y_0\left(\sqrt{\lambda}r\right)$$

e. Applying boundary conditions:

- $|R(0)| = |c_1 J_0(0) + c_2 Y_0(0)| < \infty \to c_2 = 0$ since $Y_0(0) = -\infty$
- $R(a) = c_1 J_0\left(\sqrt{\lambda}a\right) = 0$ and therefore $\sqrt{\lambda}a$ must be the zeros of ***Bessel's function of the first kind of order 0***.
- Since there is no closed form expression of these zeroes, we write λ_n where "n" denotes the n[th] zero for J_0.
- Therefore $$\sqrt{\lambda}a = \lambda_n \to \lambda = \left(\frac{\lambda_n}{a}\right)^2 \to$$ $$R_n(r) = c_n J_0\left(\frac{\lambda_n r}{a}\right) = 0$$

5. Putting it all Together to find $u(r,\theta)$

a. Recall that:

$$T'' + \lambda_n \omega^2 T = 0 \rightarrow T_n(t) = A_n \sin\left(\omega\sqrt{\lambda_n} t\right) + B_n \cos\left(\omega\sqrt{\lambda_n} t\right)$$

b. Therefore, after combining constants:

$$u(r,t) = \sum_{n=1}^{\infty} J_0\left(\frac{\lambda_n r}{a}\right)\left(A_n \sin\left(\frac{\omega\lambda_n t}{a}\right) + B_n \cos\left(\frac{\omega\lambda_n t}{a}\right)\right)$$

6. Using Initial Conditions and Orthogonality of Bessel Functions to Find Constants

a. Apply First IC: $u(r,0) = \sum_{n=1}^{\infty} B_n c_n J_0\left(\frac{\lambda_n r}{a}\right) = f(r)$

b. Multiply both sides by $J_0\left(\sqrt{\lambda_m} r\right)$ and r, then integrate for $0 < r < a \rightarrow$

$$\begin{cases} \int_0^a \left[\sum_{n=1}^{\infty} B_n J_0\left(\frac{\lambda_m r}{a}\right) J_0\left(\frac{\lambda_n r}{a}\right) r\right] dr = \int_0^a f(r) J_0\left(\frac{\lambda_n r}{a}\right) r dr \rightarrow \\ \sum_{n=1}^{\infty} B_n \left[\int_0^a J_0\left(\frac{\lambda_m r}{a}\right) J_0\left(\frac{\lambda_n r}{a}\right) r dr\right] = \int_0^a f(r) J_0\left(\frac{\lambda_n r}{a}\right) r dr \end{cases}$$

c. Using the orthogonality of Bessel functions, we know that the integral in the square brackets is non-zero if $m = n$, therefore:

$$B_n \int_0^a J_0^2\left(\frac{\lambda_n r}{a}\right) r dr = \int_0^a f(r) J_0\left(\frac{\lambda_n r}{a}\right) r dr \rightarrow \boxed{B_n = \frac{\int_0^a f(r) J_0\left(\frac{\lambda_n r}{a}\right) r dr}{\int_0^a J_0^2\left(\frac{\lambda_n r}{a}\right) r dr}}$$

d. To apply second IC we must use

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{\omega\lambda_n}{a} J_0\left(\frac{\lambda_n r}{a}\right)\left(A_n \cos\left(\frac{\omega\lambda_n}{a} t\right) - B_n \sin\left(\frac{\omega\lambda_n}{a} t\right)\right) \rightarrow$$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} A_n \frac{\omega\lambda_n}{a} J_0\left(\sqrt{\lambda_n} r\right) = g(r)$$

e. Using Orthogonality we have: $\boxed{A_n = \frac{a}{\omega\lambda_n} \frac{\int_0^a g(r) J_0\left(\frac{\lambda_n r}{a}\right) r dr}{\int_0^a J_0^2\left(\frac{\lambda_n r}{a}\right) r dr}}$

f. And the problem is done!!!

7. Example:

PDE: $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$ where $\begin{cases} 0 \le r \le 1 \\ t \ge 0 \end{cases}$ (note that $\omega = 1$)

BC: $\begin{cases} u(1,t) = 0 \\ |u(0,t)| < \infty \end{cases}$

IC: $\begin{cases} u(r,0) = (1-r) \\ \dfrac{\partial u}{\partial t}(r,0) = 0 \end{cases}$

$\lambda = (\lambda_n)^2$ where λ_n is the n^{th} zero of the J_0

$$A_n = \frac{1}{\lambda_n}\frac{\int_0^1 (0) J_0(\lambda_n r) r dr}{\int_0^1 J_0^2(\lambda_n r) r dr} = 0$$

$$B_n = \frac{\int_0^1 (1-r) J_0(\lambda_n r) r dr}{\int_0^1 J_0^2(\lambda_n r) r dr}$$

$$u(r,t) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r)\cos(\lambda_n t)$$

See Maple Demo