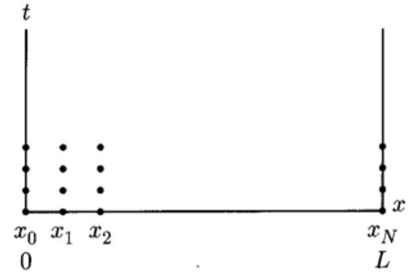
SM315 Lecture Notes
Approximating the Heat Equation with Finite Difference Equations

1. Space Time Discretization of Heat Equation in One-Dimension



a. Let: $u_j^{(m)}$ by the value of the function $u(x.t)$ at j^{th} x grid point and the m^{th} time grid point.

b. Recall the heat equation: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

c. Recall central difference equation for the second derivative in x about the point $u_j^{(m)}$:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{\Delta x^2}$$

d. Recall the forward difference equation for the first derivative in time about $u_j^{(m)}$:

$$\frac{\partial u}{\partial t} \approx \frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t}$$

e. Therefore the heat equation can be approximated by:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} \approx k\left(\frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{\Delta x^2}\right)$$

f. We can not step forward smartly in time by rewriting the equation:

$$u_j^{(m+1)} \approx u_j^{(m)} + \frac{k\Delta t}{\Delta x^2}\left(u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}\right)$$

g. As Δx and Δt become smaller, the approximation should become more accurate, however the trade off is demands on the computer assets.

h. Also there is the following stability requirement: $\frac{k\Delta t}{\Delta x^2} \leq \frac{1}{2}$

2. Example 1D-Heat Equation (Discretizing all Equations with Derivatives)

a. PDE: $\frac{\partial u}{\partial t} = .1\frac{\partial^2 u}{\partial x^2} \rightarrow u_j^{(m+1)} \approx u_j^{(m)} + \frac{.1\Delta t}{\Delta x^2}\left(u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}\right)$

b. BC: $\begin{cases} u(0,t) = .3 + .7e^{-t} \\ \frac{\partial u}{\partial x}(1,t) = 0 \rightarrow \frac{u_N^{(m)} - u_{N-1}^{(m)}}{\Delta x} = 0 \rightarrow u_N^{(m)} = u_{N-1}^{(m)} \end{cases}$

- Note: Here N is the grid point that marks the location $x = 1$. N will be dependent on Δx.
- Note: To match the accuracy of the interior grid-points, use the 3 point backward difference equation,

$$\frac{\partial u}{\partial x}(1,t) = 0 \rightarrow \frac{3u_N^{(m)} - 4u_{N-1}^{(m)} + u_{N-2}^{(m)}}{2\Delta x} = 0 \rightarrow u_N^{(m)} = \frac{4u_{N-1}^{(m)} - u_{N-2}^{(m)}}{3}$$

c. IC: $u(x,0) = \cos\left(\frac{\pi}{2}x\right)$

3. Space Time Discretization of Heat Equation in Two-Dimensions

a. Let: $u_{j,l}^{(m)}$ by the value of the function $u(x,y,t)$ at j^{th} x grid point, l^{th} y grid point, and the m^{th} time grid point.

b. Recall the heat equation: $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

c. Descritization becomes: $\frac{u_{j,l}^{(m+1)} - u_{j,l}^{(m)}}{\Delta t} \approx k\left(\frac{u_{j-1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j+1,l}^{(m)}}{\Delta x^2} + \frac{u_{j,l-1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l+1}^{(m)}}{\Delta y^2}\right)$

d. Which becomes: $\boxed{u_{j,l}^{(m+1)} \approx u_{j,l}^{(m)} + \frac{k\Delta t}{\Delta x^2}\left(u_{j-1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j+1,l}^{(m)}\right) + \frac{k\Delta t}{\Delta y^2}\left(u_{j,l-1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l+1}^{(m)}\right)}$

e. If we assume that $\Delta x = \Delta y$ the stability requirement becomes: $\frac{k\Delta t}{\Delta x^2} \le \frac{1}{4}$