SM315 Lecture Notes
Approximating the Wave Equation with FD Equations

1. Space Time Discretization of 2D Wave Equation in One-Dimension with a forcing function

a. Let: $u_{j,l}^{(m)}$ by the value of the function $u(x,y,t)$ at j^{th} x grid point, l^{th} y grid point, and the m^{th} time grid point.

b. Recall the wave equation for a rectangle with edges "pinned" down:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \omega^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + Q(x,y,t) \\ BC: \quad u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0 \\ IC: \quad u(x,y,0) = f(x,y) \;\; \& \;\; \dfrac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$

c. Descritization becomes:

$$\frac{u_{j,l}^{(m-1)} - 2u_{j,l}^{(m)} + u_{j,l}^{(m+1)}}{\Delta t^2} \approx \omega^2\left(\frac{u_{j-1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j+1,l}^{(m)}}{\Delta x^2} + \frac{u_{j,l-1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l+1}^{(m)}}{\Delta y^2}\right) + Q(x_j, y_l, t_m)$$

d. We encounter a problem with the time derivative during the first time step, i.e. $m = 1$ (note $m = 1$holds the initial condition)

$u_{j,l}^{(2)} \rightarrow$ is what we are solving for.

$u_{j,l}^{(1)} \rightarrow$ is the displacement at the initial condition

$u_{j,l}^{(0)} \rightarrow$ is a point for which I have no data

e. Resolve this by rewriting the difference equation for the time derivative,

$$\frac{u_{j,l}^{(0)} - 2u_{j,l}^{(1)} + u_{j,l}^{(2)}}{\Delta t^2} = \frac{u_{j,l}^{(2)} - u_{j,l}^{(1)}}{\Delta t^2} - \frac{1}{\Delta t}\left(\frac{u_{j,l}^{(1)} - u_{j,l}^{(0)}}{\Delta t}\right)$$

f. We recognize that $\dfrac{\partial u}{\partial t} \approx \left(\dfrac{u_{j,l}^{(1)} - u_{j,l}^{(0)}}{\Delta t}\right) = g(x,y)$ as stated by initial conditions.

g. Therefore for the first time step use:

$$\frac{u_{j,l}^{(2)} - u_{j,l}^{(1)}}{\Delta t^2} - \frac{1}{\Delta t} g(x,y) \approx \omega^2\left(\frac{u_{j-1,l}^{(1)} - 2u_{j,l}^{(1)} + u_{j+1,l}^{(1)}}{\Delta x^2} + \frac{u_{j,l-1}^{(1)} - 2u_{j,l}^{(1)} + u_{j,l+1}^{(1)}}{\Delta y^2}\right) + Q(x_j, y_l, t_m)$$

h. We now have an explicit solution for $u_{j,l}^{(2)}$ for the first time step:

$$u_{j,l}^{(2)} \approx u_{j,l}^{(1)} + g(x,y)\Delta t + \frac{\omega^2 \Delta t^2}{\Delta x^2}\left(u_{j-1,l}^{(1)} - 2u_{j,l}^{(1)} + u_{j+1,l}^{(1)}\right) + \frac{\omega^2 \Delta t^2}{\Delta y^2}\left(u_{j,l-1}^{(1)} - 2u_{j,l}^{(1)} + u_{j,l+1}^{(1)}\right) + Q(x_j, y_l, t_m)\Delta t^2$$

i. For all other time steps we use:

$$u_{j,l}^{(m+1)} \approx 2u_{j,l}^{(m)} - u_{j,l}^{(m-1)} + \frac{\omega^2 \Delta t^2}{\Delta x^2}\left(u_{j-1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j+1,l}^{(m)}\right) + \frac{\omega^2 \Delta t^2}{\Delta y^2}\left(u_{j,l-1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l+1}^{(m)}\right) + Q(x_j, y_l, t_m)\Delta t^2$$

j. If we assume that $\Delta x = \Delta y$ the stability requirement becomes: $\frac{\omega \Delta t}{\Delta x} \le \frac{1}{2}$

2. Example: Wave Equation with Forcing Function

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + xy\sin(2t) = \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \qquad i.e.\ \omega = 1 \\ BC: \ u(0,y,t) = u(2,y,t) = u(x,0,t) = u(x,1,t) = 0 \qquad i.e.\ L = 2\ \&\ H = 1 \\ IC: \ u(x,y,0) = x^2 y(2-x)(1-y)^3\ \&\ \dfrac{\partial u}{\partial t}(x,y,0) = 0 \end{cases}$$

a. Discretization becomes:

Initial Time Step (m=2)

$$u_{j,l}^{(2)} \approx u_{j,l}^{(1)} + \frac{\Delta t^2}{\Delta x^2}\left(u_{j-1,l}^{(1)} - 2u_{j,l}^{(1)} + u_{j+1,l}^{(1)}\right) + \frac{\Delta t^2}{\Delta y^2}\left(u_{j,l-1}^{(1)} - 2u_{j,l}^{(1)} + u_{j,l+1}^{(1)}\right) - x_j y_l \sin(2t_m)\ \Delta t^2$$

All Subsequent Time Steps

$$u_{j,l}^{(m+1)} \approx 2u_{j,l}^{(m)} - u_{j,l}^{(m-1)} + \frac{\Delta t^2}{\Delta x^2}\left(u_{j-1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j+1,l}^{(m)}\right) + \frac{\Delta t^2}{\Delta y^2}\left(u_{j,l-1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l+1}^{(m)}\right) - x_j y_l \sin(2t_m)\Delta t^2$$

b. Stability Requirement: $\Delta t \le \frac{1}{2}\Delta x$

c. Program for MATLAB