Score:

Name:
Period (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM315 – Quiz #2

1. Find the Fourier cosine series for $f(x)=\sin(x)$ for $0<x<\pi$.

   (a) Write out the first three non-zero terms.
   (b) Sketch the Fourier series for an infinite number of terms for $-2\pi<x<2\pi$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}(-\cos(x))\Big|_0^{+\pi}$$

$$= \frac{1}{\pi}\left(-\cos(\pi) - (-\cos(0))\right) = \frac{1}{\pi}(1+1) = 2/\pi$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{2(n+1)} - \frac{1}{2(n-1)}\right]\left((-1)^n+1\right) = \begin{cases} 0 & \text{odd terms} \\ \dfrac{-2}{(n^2-1)(\pi)} & \text{even terms} \end{cases}$$

note: $\frac{1}{n+1} - \frac{1}{n-1} = \frac{n-1-n-1}{n^2-1} = \frac{-2}{n^2-1}$

$$\Rightarrow \sin(x) \approx \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2-1}\right)\left((-1)^n+1\right)\cos(nx)$$

$n \neq 1$

$\Rightarrow$ write for even terms only

i.e. let $n=2k$ for $k=1,2,\ldots$

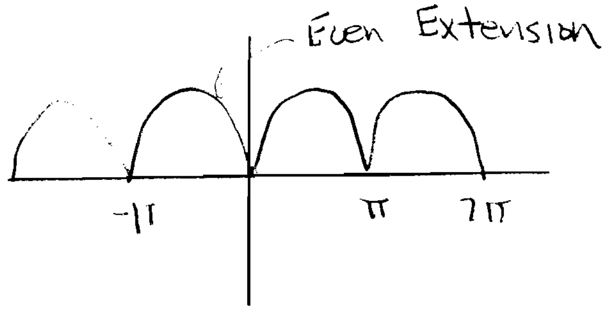
$$\Rightarrow \sin(x) \approx \frac{2}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}(2)\cos(2kx)$$

$$\Rightarrow \boxed{\sin(x) \approx \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos(2kx)}$$

(a) $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\right)\cos 2x - \frac{4}{\pi}\frac{1}{15}\cos(4x)$

$$f(x) \approx \frac{2}{\pi} - \frac{4}{3\pi}\cos(2x) - \frac{4}{15\pi}\cos(4x) \ldots$$

(b)



Alternate Solution

$$u_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

use identity $\sin(a)\cos(b) = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$

$$\Rightarrow \sin(x)\cos(nx) = \frac{1}{2}\sin(x+nx) + \frac{1}{2}\sin(x-nx)$$
$$= \frac{1}{2}\sin((1+n)(x)) + \frac{1}{2}\sin((1-n)x)$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\sin((1+n)x)\,dx + \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\sin((1-n)x)\,dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi(n+1)}\cos((n+1)x)\Big|_0^{x} + \frac{\cancel{2}}{\cancel{2}\pi(1-n)}\cos((1-n)x)\Big|_0^{\pi}$$

$$= \frac{\cancel{2}}{\cancel{2}\pi(n+1)}\left(\cos[(n+1)\pi] - 1\right) + \frac{\cancel{2}}{\cancel{2}\pi(1-n)}\left(\cos((1-n)\pi) - 1\right)$$

$\Rightarrow$ note $\cos((1-n)(\pi)) = \cos((n-1)(\pi))$

$$= \frac{1}{\pi(n+1)}\left[(-1)^{n+1} - 1\right] + \frac{1}{\pi(1-n)}\left[(-1)^{n-1} - 1\right]$$

note $(-1)^{n+1} = (-1)^{n-1}$

$$\Rightarrow \frac{1}{\pi(n+1)}\left[(-1)^{n+1} - 1\right] + \frac{1}{\pi(1-n)}\left[(-1)^{n+1} - 1\right]$$

$$= -\frac{1}{\pi(n+1)}\left[(-1)^n + 1\right] - \frac{1}{\pi(1-n)}\left[(-1)^n + 1\right]$$

$$= -\frac{1}{\pi(n+1)}\left[(-1)^n + 1\right] + \frac{1}{\pi(n-1)}\left[(-1)^n + 1\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]\left[(-1)^n + 1\right] \qquad n \neq 1$$

$$= \frac{-1}{\pi}\left[\frac{(n+1)-(n-1)}{n^2-1}\right]\left[(-1)^n+1\right] \qquad n \neq 1$$

$$= \frac{-1}{\pi}\left[\frac{2}{n^2-1}\right]\left[(-1)^n+1\right] \qquad n \neq 1$$

$$= \frac{-2}{(n^2-1)\pi}\left[(-1)^n+1\right] = -\frac{2}{\pi}\left(\frac{1}{(n^2-1)}\right)\left((-1)^n+1\right)$$

same a_n as before!