SM 315
Quiz #3

$$\boxed{\begin{array}{l} \dfrac{\partial u}{\partial t} = 5\dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x(\pi - x)^2 \end{array}}$$

$\Rightarrow \quad u = XT$

$\Rightarrow \quad XT' = 5X''T$

$\Rightarrow \quad \dfrac{T'}{5T} = \dfrac{X''}{X} = -\lambda$

$\Rightarrow \quad T' + 5\lambda T = 0 \quad \Rightarrow \quad T = a e^{-5\lambda t}$

$\quad X'' + \lambda X = 0 \quad\quad X = c_1 \sin \lambda^{1/2} t + c_2 \cos \lambda^{1/2} t$

(since $\lambda > 0$ is only non-trivial eigenvalue)

$\Rightarrow$ evaluate Boundary Conditions

$X(0) = c_2 = 0 \Rightarrow \quad X = c_1 \sin(\lambda^{1/2} x)$

$\Rightarrow X(\pi) = c_1 \sin \lambda^{1/2}\pi \Rightarrow \lambda^{1/2}\pi = n\pi \Rightarrow \lambda^{1/2} = n \Rightarrow \lambda = n^2$

$\Rightarrow \quad X_n = c_n \sin(nx) \Rightarrow \quad X = \sum_{n=1}^{\infty} c_n \sin(nx)$

$\Rightarrow u = XT = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx)$

$\Rightarrow$ evaluate initial condition

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = x(\pi - x)^2$

"Fourier Sin Series"

$$\Rightarrow \quad c_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)^2 \sin(nx)\,dx$$

$$c_n = \frac{4}{n^3}(-1)^n + \frac{8}{n^3} = \begin{cases} 4/n^3 & n \text{ odd} \\ 12/n^3 & n \text{ even} \end{cases}$$

$$\Rightarrow \boxed{u(x,t) = 4\sum_{n=1}^{\infty} \frac{1}{n^3}\left((-1)^2 + 2\right) e^{-5n^2 t} \sin(nx)}$$